Isospectral Metrics on Weighted Projective Spaces

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Abstract

The Laplace Operator on compact Riemannian manifolds naturally generalizes to compact Riemannian orbifolds and the spectrum of the resulting operator consists only of eigenvalues with finite multiplicities. The observation that the spectrum contains information about the geometry of a manifold (and, more generally, an orbifold) gave rise to a whole field of mathematics. It is an open question of so-called spectral geometry, whether a manifold and a singular orbifold can be isospectral (i.e., have the same spectrum with the same multiplicities of the eigenvalues). Given the various obstructions to the existence of such an example for the known examples of isospectral good orbifolds, this work is an attempt to shed light on the spectral geometry of bad orbifolds by giving the first examples of isospectral Riemannian metrics on bad orbifolds. In our case these are particular fixed weighted projective spaces equipped with non-trivially isospectral metrics obtained by a generalization of Schüth's version of the torus method.

Zusammenfassung

Der Laplace-Operator auf kompakten Riemannschen Mannigfaltigkeiten besitzt eine natürliche Verallgemeinerung auf kompakte Riemannsche Orbifolds und das Spektrum des so gewonnenen Operators besteht ausschließlich aus Eigenwerten endlicher Vielfachheit. Die Feststellung, dass das Spektrum Informationen über die Geometrie einer Mannigfaltigkeit (oder, allgemeiner, einer Orbifold) enthält, begründete ein ganzes Teilgebiet der Mathematik. Es ist eine offene Frage der sogenannten Spektralgeometrie, ob eine Mannigfaltigkeit und eine singuläre Orbifold isospektral sein (d.h., dasselbe Spektrum mitsamt den Vielfachheiten der Eigenwerte besitzen) können. Angesichts diverser Obstruktionen zur Existenz eines solchen Beispiels für die bekannten Beispiele isospektraler guter Orbifolds, soll diese Arbeit die Spektralgeometrie schlechter Orbifolds erhellen. Zu diesem Zweck geben wir die ersten Beispiele für isospektrale Metriken auf schlechten Orbifolds an. Diese basieren auf bestimmten gewichteten projektiven Räumen, auf denen wir mittels einer Verallgemeinerung von Schüths Version der Torus-Methode nicht-trivial isospektrale Metriken konstruieren.

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Contents

1 Introduction

An orbifold is a generalization of a smooth manifold, which is in general not locally homeomorphic to an open subset of $\mathbb{R}^n$ but to the quotient of a manifold $\tilde{U}$ by an effective action of a finite group Γ. A Riemannian metric on the orbifold is then in each orbifold chart as above given by a Γ-invariant metric on $\tilde{U}$ (and, as the charts themselves, satisfies certain compatibility conditions, of course). Given a Riemannian metric on an orbifold, it is possible to generalize the Laplace operator and it is well-known that on a compact Riemannian orbifold $(\mathcal{O}, g)$ the spectrum of the Laplacian $\Delta\colon C^\infty(\mathcal{O}) \to C^\infty(\mathcal{O})$ can be written as an infinite sequence

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \nearrow \infty$$

of eigenvalues, each repeated according to the (finite) dimension of the corresponding eigenspace ([DGGW08]). The observation that the spectrum contains geometric information like dimension, volume and certain curvature integrals gave rise to the field of Spectral Geometry which deals with the question of the degree to which the spectrum of the Laplacian determines the geometry of the given space.

Besides a vast theory on manifolds (cf. [Gor00]), the spectral geometry on orbifolds has recently received rising attention, since these provide the arguably simplest type of singular space, and it is still an open problem, whether a singular space can be isospectral to (i.e., have the same spectrum as) a manifold. Recent efforts have concentrated on isotropy groups, which in a way measure the degree of singularity of an orbifold ([Sta05],[SSW06], [RSW08]). However, all known non-trivial examples of isospectral orbifolds (also compare [Bér92], [GWW92], [Sut06], [PS08]) are good, i.e., they can be written as the quotient of a Riemannian manifold M by a discrete subgroup Γ of the isometry group of M, and the eigenspaces on the orbifold M/Γ correspond to the Γ-invariant eigenspaces on M. Since the known constructions can be seen to never yield an isospectral pair of a manifold and a singular orbifold, the considerably harder setting of bad (i.e., non-good) orbifolds deserves special attention. The isospectrality of bad orbifolds was already investigated in [ADFG08] and [GUW08], where large families of non-homeomorphic weighted projective spaces were shown to be pairwise non-isospectral. Weighted projective spaces are a generalization of complex projective space obtained by taking certain quotients of odd-dimensional spheres by S^1-actions with finite stabilizers.

In this work we now use certain weighted projective spaces and special metrics based on ideas in [Sch01] to construct isospectral metrics on bad orbifolds. Our main result is the following Theorem 5.3.1.

Theorem. *For every $n \geqslant 4$ and for all pairs (p, q) of coprime positive integers there are isospectral families of pairwise non-isometric metrics on the orbifold $\mathcal{O} = \mathcal{O}(p, q)$, a weighted projected space of dimension $2n \geqslant 8$, which is a bad orbifold for $(p, q) \neq (1, 1)$.*

This theorem generalizes a result on $\mathbb{CP}^n$ (which is the case $(p, q) = (1, 1)$ in the theorem above) from [Rüc06].

This thesis is organized as follows: In Chapter 2 we present general notions on orbifolds (as introduced in [Sat56] under the name V-manifold) which enable us to do analysis and differential geometry on these spaces. One goal is to understand orbifolds which are quotients M/G of a manifold M by an action of a compact Lie group with finite stabilizers, since this is the form in which we will write our weighted projective spaces. In particular, we construct tensor fields on the orbifold M/G from G-invariant tensor fields on M generalizing the usual construction for the case of a free G-action, where M/G becomes a manifold.

Chapter 3 summarizes basic results on the spectral geometry of compact Riemannian orbifolds including a variational characterization of eigenvalues and known constructions for isospectral (good) orbifolds. Moreover, we give some obstructions to the isospectrality between manifolds and singular orbifolds. These apply to the given constructions and thus motivate the study of bad orbifolds.

In Chapter 4 we generalize results from [Sch01] to orbifolds. The basic idea of this so-called torus method (which, in a different form, was first used in [Gor94]) is that the existence of related isometric actions of a fixed torus on two Riemannian orbifolds implies under very special conditions, that these two orbifolds are isospectral. We also show how the general criterion for non-isometry in [Sch01] easily carries over to the orbifold case.

In Chapter 5 we introduce (with n, p, q as in the theorem above) our weighted projective spaces $\mathcal{O}(p, q) = S^{2n+1}/S^1$ with the action given by $\sigma(u, v) = (\sigma^p u, \sigma^q v)$ for $\sigma \in S^1 \subset \mathbb{C}$, $u \in \mathbb{C}^{n-1}$, $v \in \mathbb{C}^2$. To apply the torus method from the preceding chapter, we first fix a space $\mathcal{O}(p, q)$. We then give a smooth action of a torus T on S^{2n+1} and certain families of 1-forms on S^{2n+1} from [Sch01], which we show to induce a smooth T-action on $\mathcal{O}(p, q)$ and families of 1-forms on $\mathcal{O}(p, q)$, respectively. To this setting we can apply the results from Chapter 4 to obtain families of isospectral metrics on $\mathcal{O}(p, q)$. For the impatient reader Subsection 5.2.2 contains an alternative isospectrality proof independent from Chapter 4 which also implies our main result but applies only to the case of isospectral families and hence misses some potential isospectral pairs. Eventually, we show that the resulting metrics are (under certain conditions) non-isometric, thus establishing our main theorem above. Moreover, inspired by [Sut06], we give isospectral metrics on quotients of our weighted projective spaces by certain finite groups.

2 Orbifold Preliminaries

2.1 General Concepts

In this work we deal with examples of orbifolds, which are a generalization of manifolds introduced by Satake ([Sat56]) and popularized by Thurston ([Thu81]). In this section we give a slightly different but essentially equivalent definition (cf. the appendix of [CR02]) and some basic statements.

Before we come to the definition of an orbifold, we need to define what we mean by charts on these structures. All manifolds in this work are second countable, Hausdorff and smooth.

Definition 2.1.1. Let X be a topological Hausdorff space which is second countable and let $U \subset X$ be a connected open subset endowed with the induced topology. An n-dimensional *orbifold chart* over U is a triple $(U, \tilde{U}/\Gamma, \pi)$, where

1. $\tilde{U}$ is a connected n-dimensional manifold.
2. Γ is a finite group acting smoothly and effectively on $\tilde{U}$ such that:

   (C) For every $\gamma \in \Gamma \setminus \{e\}$ the connected components of the fixed point set of γ have codimension at least two.
3. $\pi\colon \tilde{U} \to U$ is a continuous map invariant under Γ such that the induced map $\tilde{U}/\Gamma \to U$ is a homeomorphism with respect to the quotient topology on $\tilde{U}/\Gamma$.

Two charts $(U, \tilde{U}_i/\Gamma_i, \pi_i)$, $i = 1, 2$, over the same domain U are called *isomorphic* if there is a diffeomorphism $\lambda\colon \tilde{U}_1 \to \tilde{U}_2$ such that $\pi_2 \circ \lambda = \pi_1$.

Remark. Note that for every non-trivial element of Γ the components of the set of fixed points are closed submanifolds of $\tilde{U}$ by [Kob72] Ch. II Thm. 5.1.

A chart isomorphism λ as above is a special case of a so-called injection:

Definition 2.1.2. Let X be a topological Hausdorff space which is second countable. Let $U' \subset U$ be open and connected subsets of X and let $(U', \tilde{U}'/\Gamma', \pi')$, $(U, \tilde{U}/\Gamma, \pi)$ be two orbifold charts of the same dimension. An *injection*

$$\lambda\colon (U', \tilde{U}'/\Gamma', \pi') \to (U, \tilde{U}/\Gamma, \pi)$$

is an embedding $\lambda\colon \tilde{U}' \to \tilde{U}$ satisfying

$$\pi' = \pi \circ \lambda.$$

Remark. Given an injection $\lambda\colon \tilde{U}' \to \tilde{U}$, we will use the notation λ^{-1} for the inverse of the corestriction $\lambda\colon \tilde{U}' \to \lambda(\tilde{U}')$ although λ itself is in general not a diffeomorphism.

We now give a few statements which are convenient for the work with orbifold charts. All charts are understood to be given on a fixed second countable Hausdorff space X.

Let G be a Lie group acting smoothly on a manifold M. A connected subset $S \subset M$ is called *G-stable* if for any $g \in G$ either $gS = S$ or $gS \cap S = \emptyset$. For any G-stable subset S of M we set

$$G_S := \{g \in G;\ gS = S\} = \{g \in G;\ gS \cap S \neq \emptyset\}$$

and for $x \in M$ we set $G_x := G_{\{x\}}$.

For a proof of the following proposition see [MM03] Prop. 2.12.

Proposition 2.1.3. *1. For any injection $\lambda\colon (U_1, \tilde{U}_1/\Gamma_1, \pi_1) \to (U_2, \tilde{U}_2/\Gamma_2, \pi_2)$ the image $\lambda(\tilde{U}_1)$ is Γ_2-stable and there is a unique monomorphism $\bar{\lambda}\colon \Gamma_1 \to \Gamma_2$ with image $\Gamma_{2\lambda(\tilde{U}_1)}$ for which $\lambda(\gamma x) = \bar{\lambda}(\gamma)\lambda(x)\ \forall \gamma \in \Gamma_1, x \in \tilde{U}_1$.*

2. *The composition of two injections is an injection.*

3. *For any orbifold chart $(U, \tilde{U}/\Gamma, \pi)$ any diffeomorphism $\gamma \in \Gamma$ is an injection of $(U, \tilde{U}/\Gamma, \pi)$ into itself and $\bar{\gamma}(\gamma') = \gamma\gamma'\gamma^{-1}$ for all $\gamma' \in \Gamma$.*

4. *If $\lambda, \mu\colon (U_1, \tilde{U}_1/\Gamma_1, \pi_1) \to (U_2, \tilde{U}_2/\Gamma_2, \pi_2)$ are two injections between the same orbifold charts, there exists a unique $\gamma \in \Gamma_2$ with $\lambda = \gamma \circ \mu$.*

Moreover, it is not hard to show that for injections λ from π_1 to π_2 and μ from π_2 to π_3 one has

$$\overline{\mu \circ \lambda} = \bar{\mu} \circ \bar{\lambda}.$$

Theorem 2.1.4. *Let $(U, \tilde{U}/\Gamma, \pi)$ be a chart, and let U' be a connected open subset of $U \subset X$. Then there is a chart $(U', \tilde{U}'/\Gamma', \pi')$ over U' such that there exists an injection from $(U', \tilde{U}'/\Gamma', \pi')$ into $(U, \tilde{U}/\Gamma, \pi)$. Any two charts over U' from which there is an injection into $(U, \tilde{U}/\Gamma, \pi)$ are isomorphic.*

Proof. (see [CR02] 4.1.) □

Definition 2.1.5. The unique isomorphism class from the preceding theorem is called the isomorphism class of charts over U' *induced* by $(U, \tilde{U}/\Gamma, \pi)$.

Moreover, it is not hard to show that with $U' \subset U$ as in the theorem above, isomorphic charts over U induce the same isomorphism class of charts over U' (cf. [Wei07] Cor. 2.7).

Definition 2.1.6. Let $(U, \tilde{U}/\Gamma, \pi)$ and $(U', \tilde{U}'/\Gamma', \pi')$ be orbifold charts and let $x \in U \cap U'$. The two charts are called *equivalent at x* if there is an open connected subset $U'' \subset U \cap U'$ containing x such that the two isomorphism classes of charts on U'' induced by $(U, \tilde{U}/\Gamma, \pi)$ and $(U', \tilde{U}'/\Gamma', \pi')$ are identical. In this case we write $\pi \sim_x \pi'$.

Note that the equivalence relation $\sim_x$ above defines just a germ of charts around x.

Definition 2.1.7. An *orbifold atlas* $\mathfrak{A}$ of dimension n on a second countable Hausdorff space X is a set $\mathfrak{A} = \{(U_\alpha, \tilde{U}_\alpha/\Gamma_\alpha, \pi_\alpha)\}_{\alpha \in I(\mathfrak{A})}$ of n-dimensional orbifold charts such that

1. $\bigcup_\alpha U_\alpha = X$
2. If $x \in U_\alpha \cap U_\beta$ then $\pi_\alpha \sim_x \pi_\beta$.

Two orbifold atlases are called *equivalent* if their union is again an orbifold atlas.

Since for every $x \in \mathcal{O}$ the relation $\sim_x$ is an equivalence relation, it is straightforward to show:

Lemma 2.1.8. *Let X be a second countable Hausdorff space with an orbifold atlas $\mathfrak{A} = \{(U_\alpha, \tilde{U}_\alpha/\Gamma_\alpha, \pi_\alpha)\}_{\alpha \in I(\mathfrak{A})}$. Then there is a unique maximal atlas on X containing $\mathfrak{A}$.*

Definition 2.1.9. An n-dimensional *orbifold* is a pair $\mathcal{O} = (X, \{(U_\alpha, \tilde{U}_\alpha/\Gamma_\alpha, \pi_\alpha)\}\mathfrak{A})$ of a second countable Hausdorff space X (called the *underlying space*) and a maximal n-dimensional orbifold atlas (called the *orbifold structure*) on X or equivalently a second countable Hausdorff space X with an equivalence class of orbifold atlases on X.

An *oriented* orbifold is a pair $\mathcal{O} = (X, \mathfrak{A})$ such that $\mathfrak{A}$ is a maximal oriented orbifold atlas, i.e. for every $(U, \tilde{U}/\Gamma, \pi) \in \mathfrak{A}$ the manifold $\tilde{U}$ is oriented and the elements of Γ preserve the orientation. Moreover, all injections can be chosen to be orientation-preserving.

Remark. Given an orbifold $\mathcal{O} = (X, \mathfrak{A})$, the word chart will usually refer to a chart in $\mathfrak{A}$. Only where there is no orbifold structure given on a topological space X, the term chart still stands for the general meaning from Definition 2.1.1.

Now let $\mathcal{O}$ be an orbifold, $x \in \mathcal{O}$ and let $(U, \tilde{U}/\Gamma, \pi)$ be a chart with $x \in U$. For $\tilde{x} \in \pi^{-1}(x) \subset \tilde{U}$ let $\Gamma_{\tilde{x}} = \{\gamma \in \Gamma;\ \gamma\tilde{x} = \tilde{x}\}$ be the isotropy group (or stabilizer) of $\tilde{x}$ under the action of Γ. For another $\tilde{x}' \in \pi^{-1}(x)$ there is $\gamma \in \Gamma$ such that $\gamma\tilde{x} = \tilde{x}'$. Then $\Gamma_{\tilde{x}'} = \gamma\Gamma_{\tilde{x}}\gamma^{-1}$; i.e., the isotropy groups over x in this fixed chart form a well-defined conjugacy class of subgroups of Γ. More generally, one has

Proposition 2.1.10. *Let $x \in \mathcal{O}$ and let $(U_i, \tilde{U}_i/\Gamma_i, \pi_i)$, $i = 1, 2$, be charts with $x \in U_i$. If $\tilde{x}_i \in \pi_i^{-1}(x)$, then the groups $\Gamma_{1\tilde{x}_1}$ and $\Gamma_{2\tilde{x}_2}$ are isomorphic.*

Proof. [Bor92] □

Definition 2.1.11. Let $\mathcal{O}$ be an orbifold, $x \in \mathcal{O}$, let $(U, \tilde{U}/\Gamma, \pi)$ be a chart around x and $\tilde{x} \in \pi^{-1}(x)$. The isomorphism class of $\Gamma_{\tilde{x}}$ is called the *isotropy of* x and is denoted by $\mathrm{Iso}(x)$. If $\mathrm{Iso}(x)$ is trivial, then x is called *regular*. If $\mathrm{Iso}(x)$ is non-trivial, then x is called *singular*. The set of regular points in $\mathcal{O}$ is denoted by $\mathcal{O}^{\mathrm{reg}}$.

Remark 2.1.12. Note that for every chart $(U, \tilde{U}/\Gamma, \pi)$ on $\mathcal{O}$ the set $U^{\mathrm{reg}} = \{x \in U;\ \mathrm{Iso}(x) \text{ trivial}\}$ is open and dense in U because Γ is finite and acts effectively and hence $\tilde{U}^{\mathrm{reg}} := \pi^{-1}(U^{\mathrm{reg}})$ is open and dense in $\tilde{U}$ (cf. [Kaw91] Exercise 4.5). Hence $\mathcal{O}^{\mathrm{reg}}$ is open and dense in $\mathcal{O}$. Moreover, $\tilde{U}^{\mathrm{reg}}$ is connected for every chart on $\mathcal{O}$ by Lemma 2.1.13 and condition (C). This implies that if $\mathcal{O}$ is connected, so is $\mathcal{O}^{\mathrm{reg}}$ because $U^{\mathrm{reg}} \subset U$ is connected for every chart on $\mathcal{O}$ and one can apply an argument similar to the one in the proof of that lemma to find curves connecting two arbitrary points in $\mathcal{O}^{\mathrm{reg}}$.

Lemma 2.1.13. *Let M be a connected manifold and let $N \subset M$ be a (not necessarily connected) submanifold of codimension at least 2. Then $M \setminus N$ is connected.*

Proof. Set $m = \dim M$, $n = \dim N$. Let $x, y \in M \setminus N$. There is a curve $\gamma\colon [a,b] \to M$ with $\gamma(a) = x, \gamma(b) = y$. Choose charts $\{(x_i, U_i)\}_{i=0}^{k-1}$ on M such that

- $\gamma([a,b]) \subset \bigcup_i U_i$ and each U_i is connected,
- $x_i(U_i) = U_i' \times U_i''$ with U_i' an open subset of $\mathbb{R}^n$ and U_i'' an open subset of $\mathbb{R}^{m-n}$ containing 0.
- $x_i(N \cap U_i) \subset U_i' \times \{0\}$,
- there is a partition $a = t_0 < t_1 < \ldots < t_k = b$ such that $\gamma(t_i) \in U_{i-1} \cap U_i$ for $i \in \{1, \ldots, k-1\}$ and $\gamma(a) \in U_0$, $\gamma(b) \in U_{k-1}$.

Note that for every $i \in \{0, \ldots, k-1\}$ the set U_i' is connected and (since U_i'' is connected and $m - n \geqslant 2$) so is $U_i'' \setminus \{0\}$ and therefore $U_i \setminus N = x_i^{-1}(U_i' \times U_i'' \setminus \{0\})$.

For $i \in \{1, \ldots, k-1\}$ choose $y_i \in U_{i-1} \cap U_i \setminus N$ and let $y_0. = x, y_k := y$. Then for each $i \in \{1, \ldots, k\}$ choose a path in $U_{i-1} \setminus N$ joining y_{i-1} to y_i. The concatenation of these paths gives a (not necessarily smooth) curve from x to y in $M \setminus N$. □

Lemma 2.1.14. *Let M be a connected manifold and let $(U, \tilde{U}/\Gamma, \pi)$ be a chart on an arbitrary orbifold. If $f_1, f_2\colon M \to \tilde{U}$ are two smooth maps such that f_1 a submersion onto its image and $\pi \circ f_1 = \pi \circ f_2$ then there is a unique $\gamma \in \Gamma$ such that $\gamma \circ f_1 = f_2$.*

Proof. First note that $\pi^{\mathrm{reg}} := \pi_{|\tilde{U}^{\mathrm{reg}}}\colon \tilde{U}^{\mathrm{reg}} \to U^{\mathrm{reg}}$ is a covering map. Recall that $\tilde{U}^{\mathrm{reg}}$ is dense in $\tilde{U}$. This implies that since the submersion f_1 locally has the form $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) \mapsto (x_1, \ldots, x_n)$ ($m = \dim M$, $n = \dim \tilde{U}$), the preimage $V := f_1^{-1}(\tilde{U}^{\mathrm{reg}}) = f_2^{-1}(\tilde{U}^{\mathrm{reg}})$ is dense in M.

Moreover, V is connected: First, for $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ set $\tilde{U}^\gamma := \{y \in \tilde{U};\ \gamma y = y\}$ and for $i = 0, \ldots, n-2$ let $\tilde{U}_i^\gamma$ denote the (possibly empty) union of the components of $\tilde{U}^\gamma$ of dimension i. Then each $f_1^{-1}(\tilde{U}_i^\gamma)$ is a closed submanifold of M of codimension $n - i \geqslant 2$. Since Γ is finite, Lemma 2.1.13 implies that

$$V = M \setminus \bigcup_{\substack{i=0,\ldots,n-2 \\ \gamma \in \Gamma \setminus \{\mathrm{id}\}}} f_1^{-1}(\tilde{U}_i^\gamma)$$

is connected. Hence we can apply the unique lifting property to the lifts $f_{1|V}$ and $f_{2|V}$ of $\pi \circ f_{1|V}$ with respect to the covering π^{reg} to show that there is a unique $\gamma \in \Gamma$ such that $\gamma \circ f_{1|V} = f_{2|V}$. Since V is dense in M, we conclude that $\gamma \circ f_1 = f_2$ on all of M. □

Definition 2.1.15. Let $\mathcal{O}_1$, $\mathcal{O}_2$ be orbifolds. A *smooth map* is a continuous map $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ between the underlying spaces such that for every $x \in \mathcal{O}_1$ there is a chart $(U_1, \tilde{U}_1/\Gamma_1, \pi_1)$ around x, a chart $(U_2, \tilde{U}_2/\Gamma_2, \pi_2)$ around $f(x)$, a smooth map $\tilde{f} \in C^\infty(\tilde{U}_1, \tilde{U}_2)$ and a homomorphism $\Theta\colon \Gamma_1 \to \Gamma_2$ such that $f \circ \pi_1 = \pi_2 \circ \tilde{f}$ and $\tilde{f} \circ \gamma = \Theta(\gamma) \circ \tilde{f}\ \forall \gamma \in \Gamma_1$; i.e., the following diagram commutes.

$$
\begin{tikzcd}
\tilde{U}_1 \arrow[r, "\tilde{f}"] \arrow[d] \arrow[ddd, bend right=60, "\pi_1"'] & \tilde{U}_2 \arrow[d] \arrow[ddd, bend left=60, "\pi_2"] \\
\tilde{U}_1/\Gamma_1 \arrow[r] \arrow[dd, "\approx"] & \tilde{U}_2/\Theta(\Gamma_1) \arrow[d] \\
 & \tilde{U}_2/\Gamma_2 \arrow[d, "\approx"] \\
U_1 \arrow[r, "f"] & U_2
\end{tikzcd}
$$

Remark 2.1.16. It is pretty straightforward to check that the composition of smooth orbifold maps is smooth (cf. [Wei07] Lemma 2.19).

Moreover, note that given f and two charts π_1, π_2 as above and, moreover, a submersion $\tilde{f}$ such that $f \circ \pi_1 = \pi_2 \circ \tilde{f}$, Lemma 2.1.14 (applied to $f_1 := \tilde{f}$ and $f_2 := \tilde{f} \circ \gamma$) implies the existence of a unique map $\Theta \colon \Gamma_1 \to \Gamma_2$ satisfying $\tilde{f} \circ \gamma = \Theta(\gamma) \circ f$. This relation then implies that Θ is a homomorphism. Hence if $\tilde{f}$ in the definition above is a submersion satisfying $f \circ \pi_1 = \pi_2 \circ \tilde{f}$, then the homomorphism Θ automatically exists and is unique.

Although in general there are no tangent vector spaces (and hence no linear differential maps) on an orbifold, we can at least define the notion of rank in each orbifold point.

Definition 2.1.17. Let $f \colon \mathcal{O}_1 \to \mathcal{O}_2$ be a smooth map between orbifolds. For each $x \in \mathcal{O}_1$ define the *rank* of f in x to be the rank of a local lift $\tilde{f}$ in a point $\tilde{x}$ mapping to x under a chart around x. f is called *submersion* if its rank is everywhere equal to $\dim \mathcal{O}_2$.

Note that the definition of an orbifold atlas implies that the rank is well-defined. Moreover, the proof of the smoothness of the composition of two smooth maps mentioned above also implies that the composition of two submersions is a submersion.

Definition 2.1.18. A *diffeomorphism* between two orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ is a homeomorphism $f \colon \mathcal{O}_1 \to \mathcal{O}_2$ between the underlying spaces such that both f and f^{-1} are smooth.

Lemma 2.1.19. *A homeomorphism $f \colon \mathcal{O}_1 \to \mathcal{O}_2$ is a diffeomorphism if and only if around every $x \in \mathcal{O}_1$ there are charts and a lift $\tilde{f}$ as in Definition 2.1.15 such that $\tilde{f}$ is a diffeomorphism.*

Proof. First assume that around each $x \in \mathcal{O}_1$ the lift $\tilde{f}$ can be chosen to be a diffeomorphism. Then f is obviously smooth. Moreover f^{-1} is also smooth because for $x \in \mathcal{O}_1$ the homomorphism $\Theta \colon \Gamma_1 \to \Gamma_2$ corresponding to $\tilde{f}$ is given by $\Theta(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$, in particular it is an isomorphism. Hence the pair $(\tilde{f}^{-1}, \Theta^{-1})$ satisfies the conditions in Definition 2.1.15 with respect to f^{-1} around $f(x)$.

Now assume that $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ is a diffeomorphism according to Definition 2.1.18 and let $x \in \mathcal{O}_1$. Since f is smooth, we can find a neighbourhood U_1 of x over which f lifts. Analogously $g := f^{-1}$ lifts over a neighbourhood U_2 of $f(x)$. Then let V_1 be the component of $U_1 \cap f^{-1}(U_2)$ containing x and set $V_2 := f(V_1) \subset U_2$. By Theorem 2.1.4 there are charts $(V_i, \tilde{V}_i/\Gamma_i, \pi_i), i = 1,2$, and, moreover, there are $\tilde{f} \in C^\infty(\tilde{V}_1, \tilde{V}_2), \tilde{g} \in C^\infty(\tilde{V}_2, \tilde{V}_1)$ such that the following diagram commutes.

$$\begin{tikzcd}
\tilde{V}_1 \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"] & \tilde{V}_2 \arrow[r, "\tilde{g}"] \arrow[d, "\pi_2"] & \tilde{V}_1 \arrow[d, "\pi_1"] \\
V_1 \arrow[r, "f"] & V_2 \arrow[r, "g"] & V_1
\end{tikzcd}$$

But $\pi_1 \circ \tilde{g} \circ \tilde{f} = \pi_1$ implies $\tilde{g} \circ \tilde{f} \in \Gamma_1$ by Lemma 2.1.14 with $f_1 := \mathrm{Id}_{\tilde{V}_1}, f_2 := \tilde{g} \circ \tilde{f}$ (or, alternatively, [MM03] Lemma 2.11). Analogously, $\tilde{f} \circ \tilde{g} \in \Gamma_2$, hence $\tilde{f}$ is a diffeomorphism. $\square$

This characterization of diffeomorphisms immediately yields the following lemma.

Lemma 2.1.20. *Let $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ be a diffeomorphism and $x \in \mathcal{O}_1$. Then* $\mathrm{Iso}(x) = \mathrm{Iso}(f(x))$.

In later sections we will need to work with product orbifolds.

Definition 2.1.21. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two smooth orbifolds with underlying spaces X_1, X_2 and with atlases $\{(U_\alpha, \tilde{U}_\alpha/\Gamma_\alpha, \pi_\alpha)\}_\alpha$ and $\{(U_\beta, \tilde{U}_\beta/\Gamma_\beta, \pi_\beta)\}_\beta$, respectively. A smooth atlas on the *product orbifold* $\mathcal{O}_1 \times \mathcal{O}_2$ with underlying space $X_1 \times X_2$ is given by

$$\left\{\left(U_\alpha \times U_\beta, (\tilde{U}_\alpha \times \tilde{U}_\beta)/(\Gamma_\alpha \times \Gamma_\beta), \pi_\alpha \times \pi_\beta\right)\right\}_{\alpha,\beta}.$$

Remark. One easily checks that this is indeed an orbifold atlas. To see this, one uses injections of the form $\lambda_\alpha \times \lambda_\beta$.

Moreover, note that $\mathrm{Iso}^{\mathcal{O}_1 \times \mathcal{O}_2}(x,y) = \mathrm{Iso}^{\mathcal{O}_1}(x) \times \mathrm{Iso}^{\mathcal{O}_2}(y)$ for all $(x,y) \in \mathcal{O}_1 \times \mathcal{O}_2$.

2.2 Quotient Orbifolds

Assume we are given a manifold M and a Lie group G acting smoothly on M from the left. In the main part of this section we will give conditions under which the quotient M/G naturally becomes an orbifold. Note that the charts given in the proof of Theorem 2.2.1 will later be used to define certain structures on the orbifold M/G.

Moreover, in Theorem 2.2.4 we will show that given an orbifold $\mathcal{O}$ and a finite group G acting smoothly and effectively on $\mathcal{O}$ there is a canonical orbifold structure on the quotient $\mathcal{O}/G$.

2.2.1 Quotients of Manifolds by Connected Lie Groups

In the proof of the following theorem and in later sections we need some terminology from foliation theory. Recall that a foliation is a special kind of partition of a manifold into immersed submanifolds of constant dimension, so-called leaves (cf. [CC00]).

An action of a compact Lie group G on a manifold M is called *almost free* if the stabilizer $G_{\tilde{x}} = \{g \in G;\ g\tilde{x} = \tilde{x}\}$ is finite for every $\tilde{x} \in M$. It can be shown that in this situation (under a condition on the codimension of the fixed point sets) the quotient M/G becomes an orbifold. We follow basically the proof of [Mol88] Thm. 3.8. Also compare [MM03] Thm. 2.15.

Theorem 2.2.1. *Let M be a smooth manifold and G a compact group of diffeomorphisms acting effectively and almost freely on M such that the fixed point set of each element of $G \setminus \{e\}$ has codimension at least $\dim G + 2$. Let $P\colon\ M \to M/G$ denote the quotient map and let M/G be endowed with the quotient topology. Then there is – up to diffeomorphism – a unique orbifold structure on M/G of dimension $\dim M - \dim G$ such that $M \to M/G$ is a submersion and for every orbifold $\mathcal{O}$ a map $f\colon\ M/G \to \mathcal{O}$ is a submersion onto its image if and only if $f \circ P$ is a submersion onto its image. For this structure we have that for every $\tilde{x} \in M$ the group $G_{\tilde{x}}$ gives the orbifold isotropy in $x := P(\tilde{x}) \in M/G$.*

Proof. Uniqueness is clear because the identity on M/G gives a diffeomorphism between two orbifold structures satifying the properties above.

To show existence note that for every $\tilde{x} \in M$ the orbit $G\tilde{x}$ is an embedded submanifold of M. It has dimension $\dim G$ since the immersion $G \ni g \mapsto g\tilde{x} \in M$ induces a diffeomorphism between $G/G_{\tilde{x}}$ and $G\tilde{x}$. These orbits give a foliation on M into manifolds of dimension $\dim G$: Since $[X_1^*, X_2^*] = -[X_1, X_2]^*\ \forall X_1, X_2 \in \mathfrak{g}$ (where $X_i^*(\tilde{x}) = \frac{d}{dt}_{|t=0} \exp(tX_i)\tilde{x}$ denotes a fundamental vector field), the distribution given by the vectors tangent to the G-orbits is involutive. By the Frobenius Theorem the orbits form a foliation. The quotient space M/G is Hausdorff and second countable by elementary arguments (see the proof of [Mol88] Thm. 3.8).

Charts on M/G: Now choose a G-invariant Riemannian metric on M (which is possible since G is compact) and introduce the following notation: For a submanifold W of M let $NW \subset TM$ denote the normal bundle over W with respect to the given metric on M and let $\pi_W^\perp\colon\ NW \to W$ denote the bundle projection. Then for $\varepsilon > 0$ set

$$
\begin{aligned}
N_{\tilde{x}}^\varepsilon(W) &= \{X \in N_{\tilde{x}}W;\ \|X\| < \varepsilon\} \text{ (for } \tilde{x} \in W),\\
N^\varepsilon(W) &= \{X \in NW;\ \|X\| < \varepsilon\},\\
B^\varepsilon(W) &= \{\tilde{x} \in M;\ \operatorname{dist}(\tilde{x}, W) < \varepsilon\}.
\end{aligned}
$$

Given $\tilde{x} \in M$ we want to construct an orbifold chart around $x := P(\tilde{x})$. To this end choose $\varepsilon > 0$ such that

(1) $\exp_{|N^\varepsilon(G\tilde{x})}\colon\ N^\varepsilon(G\tilde{x}) \to B^\varepsilon(G\tilde{x})$ is a diffeomorphism, where exp denotes the geodesic exponential map.

(2) If $\psi := \pi^{\perp}_{G\tilde{x}} \circ (\exp_{|N^{\varepsilon}(G\tilde{x})})^{-1} \colon B^{\varepsilon}(G\tilde{x}) \to G\tilde{x}$ denotes the orthogonal projection onto $G\tilde{x}$, we have for every $\tilde{y} \in G\tilde{x}$:

  (a) $\psi^{-1}(\tilde{y})$ is transverse to the given foliation by G-orbits,

  (b) $\tilde{y}$ has an open neighbourhood V in $G\tilde{x}$ such that there is a manifold S and a surjective submersion $h\colon \psi^{-1}(V) \to S$ for which

    (i) every preimage $h^{-1}(s)$ $(s \in S)$ is connected and the restriction of the given foliation to $\psi^{-1}(V)$ is given precisely by these preimages (where a leaf of the restriction is by definition given by a connected component of the intersection of an orbit with $\psi^{-1}(V)$),

    (ii) for every $\tilde{y}' \in V$ the restriction $h_{|\psi^{-1}(\tilde{y}')}\colon \psi^{-1}(\tilde{y}') \to S$ is a diffeomorphism.

Such an ε exists by [Mol88], Lemma 3.7.

Now let us define a chart $(U, \tilde{U}/\Gamma, \pi)$ around $x := P(\tilde{x})$: Set $U := B^{\varepsilon}(G\tilde{x})/G \subset M/G$, $\tilde{U} := \psi^{-1}(\tilde{x}) \subset M$, $\Gamma := G_{\tilde{x}}$ and $\pi := P_{|\tilde{U}}$. First note that P is open because the preimage of a set $P(\Omega)$ with Ω an open subset of M is $\bigcup_{g\in G} g\Omega$ and hence open. Therefore U is open. Moreover, note that $\psi^{-1}(\tilde{x})$ is $G_{\tilde{x}}$-invariant because $N^{\varepsilon}_{\tilde{x}} := N^{\varepsilon}_{\tilde{x}}(G\tilde{x}) = (\exp_{|N^{\varepsilon}(G\tilde{x})})^{-1}(\psi^{-1}(\tilde{x}))$ is (where we let G act on TM by differentials) and exp is G-equivariant.

We denote the quotient of the free Γ-action $\gamma \circ (g, X) := (g\gamma^{-1}, \gamma_* X)$ on $G \times N^{\varepsilon}_{\tilde{x}}$ by $G \times_{\Gamma} N^{\varepsilon}_{\tilde{x}}$ and let G act on it from the left by multiplication in the first component. Composing the G-equivariant diffeomorphisms

$$G \times_{\Gamma} N^{\varepsilon}_{\tilde{x}} \ni [g, X] \mapsto g_* X \in N^{\varepsilon}(G\tilde{x})$$

and $\exp_{|N^{\varepsilon}(G\tilde{x})}\colon N^{\varepsilon}(G\tilde{x}) \to B^{\varepsilon}(G\tilde{x})$ and factoring out G we obtain a homeomorphism

$$N^{\varepsilon}_{\tilde{x}}/\Gamma \to N^{\varepsilon}(G\tilde{x})/G \to B^{\varepsilon}(G\tilde{x})/G.$$

Since $(\exp_{|N^{\varepsilon}_{\tilde{x}}})^{-1}\colon \psi^{-1}(\tilde{x}) \to N^{\varepsilon}_{\tilde{x}}$ is Γ-equivariant, composing the induced homeomorphism $\psi^{-1}(\tilde{x})/\Gamma \to N^{\varepsilon}_{\tilde{x}}/\Gamma$ with the homeomorphism above gives a homeomorphism

$$\tilde{U}/\Gamma = \psi^{-1}(\tilde{x})/\Gamma \to B^{\varepsilon}(G\tilde{x})/G = U.$$

(Note that this homeomorphism simply maps the Γ-equivalence class of a point $\tilde{y}$ in $\psi^{-1}(\tilde{x})$ to $[\tilde{y}] \in B^{\varepsilon}(G\tilde{x})/G$.)

Moreover, the Γ-action on $\tilde{U}$ satisfies condition (C) from Definition 2.1.1 because the fixed point set of any $g \in G\setminus\{e\}$ in $\tilde{U}$ has codimension at least $\dim \tilde{U} - \dim M + \dim G + 2 = 2$ by assumption.

Hence $(U, \tilde{U}/\Gamma, \pi)$ as chosen above is a chart on the topological space M/G.

Chart compatibility: To show that the construction above gives an orbifold, we have to prove that those charts are compatible. Let $(U_i, \tilde{U}_i/\Gamma_i, \pi_i)$, $i = 1, 2$ be charts around

$P(\tilde{x}_i)$ with $\tilde{x}_i \in \tilde{U}_i$ as constructed above such that $U_1 \cap U_2 \neq \emptyset$ and denote the corresponding objects from (1), (2) by ε_i, ψ_i. Let $y \in U_1 \cap U_2 \subset M/G$ be arbitrary. For $i = 1, 2$ choose $\tilde{y}_i \in \pi_i^{-1}(y) \subset \psi_i^{-1}(\tilde{x}_i)$, let V_i be a neighbourhood of $\tilde{x}_i$ in $G\tilde{x}_i$ as in (2)(b) and denote a corresponding submersion by $h_i\colon\ \psi_i^{-1}(V_i) \to S_i$. Since $P(\tilde{y}_1) = y = P(\tilde{y}_2)$, there is $g \in G$ such that $g\tilde{y}_1 = \tilde{y}_2$. Then choose $\delta > 0$ such that (1), (2) hold with respect to δ and $\tilde{y}_1$ and such that

$$B_\delta(G\tilde{y}_1) \subset B_{\varepsilon_1}(G\tilde{x}_1) \cap B_{\varepsilon_2}(G\tilde{x}_2).$$

Moreover, assume that for the orthogonal projection

$$\hat{\psi} := \pi^{\perp}_{G\tilde{y}_1} \circ (\exp_{|N^\delta(G\tilde{y}_1)})^{-1}\colon\ B^\delta(G\tilde{y}_1) \to G\tilde{y}_1$$

we have

$$\hat{\psi}^{-1}(\tilde{y}_1) \subset \psi_1^{-1}(V_1) \text{ and } g\hat{\psi}^{-1}(\tilde{y}_1) \subset \psi_2^{-1}(V_2).$$

Since $\hat{\psi}^{-1}(\tilde{y}_1)$ is transverse to the G-orbits, $h_{1|\hat{\psi}^{-1}(\tilde{y}_1)}$ and $h_{2|g\hat{\psi}^{-1}(\tilde{y}_1)}$ are local diffeomorphisms. Choose δ so small that these restrictions become embeddings.

This gives a chart $\hat{\pi} := P_{|\hat{\psi}^{-1}(\tilde{y}_1)}$ with corresponding group $\hat{\Gamma} := G_{\tilde{y}_1}$ over $B^\delta(G\tilde{y}_1)/G \ni y$. Note that the restrictions

$$h_{i|\psi_i^{-1}(\tilde{x}_i)}\colon\ \psi_i^{-1}(\tilde{x}_i) \to S_i$$

are diffeomorphisms by (2)(b)(ii). Hence the embedding

$$h_{1|\hat{\psi}^{-1}(\tilde{y}_1)}\colon\ \hat{\psi}^{-1}(\tilde{y}_1) \to S_1$$

gives an injection

$$\lambda_1 := \left(h_{1|\psi_1^{-1}(\tilde{x}_1)}\right)^{-1} \circ h_{1|\hat{\psi}^{-1}(\tilde{y}_1)}\colon\ \hat{\psi}^{-1}(\tilde{y}_1) \to \psi_1^{-1}(\tilde{x}_1)$$

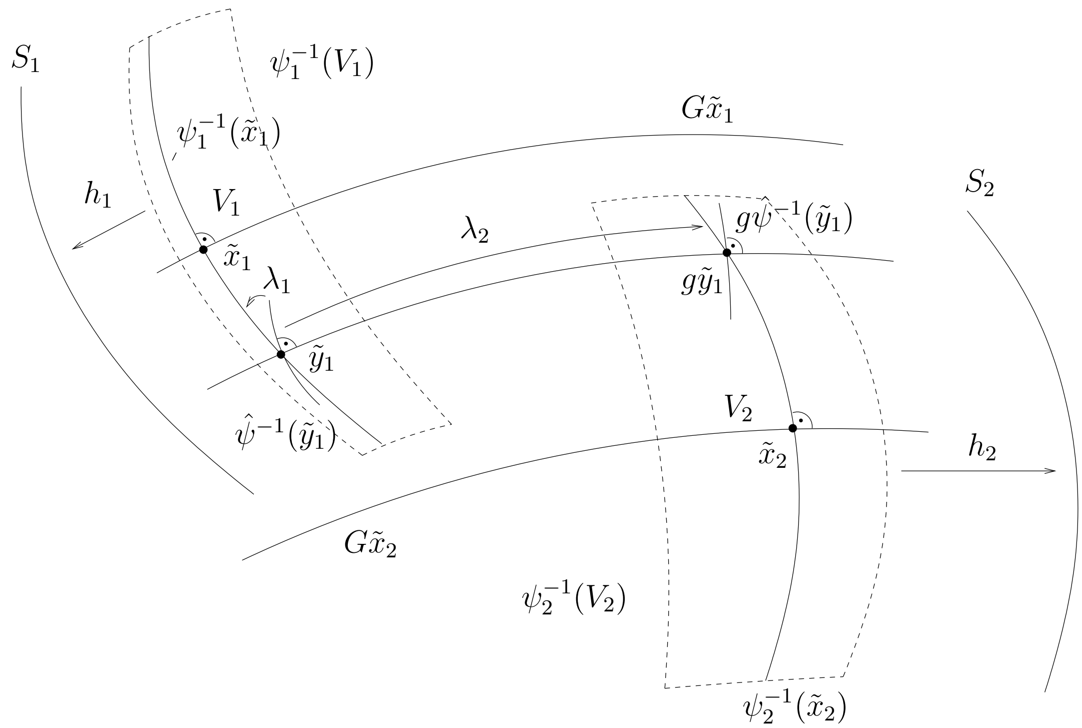
from $\hat{\pi}$ into π_1 and the embedding

$$h_{2|g\hat{\psi}^{-1}(\tilde{y}_1)} \circ g\colon\ \hat{\psi}^{-1}(\tilde{y}_1) \to S_2$$

gives an injection

$$\lambda_2 := \left(h_{2|\psi_2^{-1}(\tilde{x}_2)}\right)^{-1} \circ h_2 \circ g_{|\hat{\psi}^{-1}(\tilde{y}_1)}\colon\ \hat{\psi}^{-1}(\tilde{y}_1) \to \psi_2^{-1}(\tilde{x}_2)$$

from $\hat{\pi}$ into π_2. (λ_1 and λ_2 are injections because both $(h_{i|\psi_i^{-1}(\tilde{x}_i)})^{-1} \circ h_i$ and g leave orbits invariant and hence $P \circ \lambda_i = P$ on $\hat{\psi}^{-1}(\tilde{y}_1)$.) Hence we have shown that the charts defined above induce an orbifold structure on M/G. By construction, the isotropy of a point $P(\tilde{x})$ is indeed given by $G_{\tilde{x}}$.



Properties of P: To see that $P\colon M \to M/G$ is a submersion with M/G endowed with this structure, let $\tilde{x}$ be an arbitrary point in M. Let $\varepsilon > 0$ satisfy Conditions (1) and (2) above with respect to $\tilde{x}$, let V be a neighbourhood of $\tilde{x}$ in $G\tilde{x}$ as in (2)(b) and let $h\colon \psi^{-1}(V) \to S$ be a corresponding submersion. Then $h_{|\psi^{-1}(\tilde{x})}$ is a diffeomorphism and

$$\phi := \left(h_{|\psi^{-1}(\tilde{x})}\right)^{-1} \circ h\colon \psi^{-1}(V) \to \psi^{-1}(\tilde{x}) \tag{2.1}$$

is a submersion of manifolds. Since $P \circ \phi = P$ on $\psi^{-1}(V)$, the following diagram commutes.

$$\begin{array}{ccccc} & \psi^{-1}(V) & \xrightarrow{\phi} & \psi^{-1}(\tilde{x}) & \\ & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi} & \\ M \supset & \psi^{-1}(V) & \xrightarrow{P} & B^{\varepsilon}(G\tilde{x})/G & \subset M/G \end{array}$$

Hence P is an orbifold submersion.

Since the composition of submersions automatically is a submersion, it remains to show that a map $f\colon M/G \to \mathcal{O}$ is a submersion if $f \circ P$ is a submersion. So suppose that $f \circ P$ is a submersion, let $x \in M/G$ and choose $\widetilde{x} \in P^{-1}(x)$. Then let $\varepsilon > 0$ and the corresponding ψ, V be as above, where we choose ε and V so small that there is a chart $(U_2, \widetilde{U}_2/\Gamma_2, \pi_2)$ around $f(x)$ in $\mathcal{O}$ together with a submersion $\widetilde{f \circ P}\colon \psi^{-1}(V) \to \widetilde{U}_2$ such that $\pi_2 \circ \widetilde{f \circ P} = f \circ P_{|\psi^{-1}(V)}$. As usual, set $U_1 := B^{\varepsilon}(G\widetilde{x})/G$, $\widetilde{U}_1 := \psi^{-1}(\widetilde{x})$, $\Gamma_1 := G_{\tilde{x}}$ and $\pi_1 := P_{|\psi^{-1}(\widetilde{x})}$

Now note that since Γ_2 is finite and $f \circ P$ is constant on orbits, $\widetilde{f \circ P}$ is (by continuity) also locally constant on the intersection of $\psi^{-1}(V)$ with each orbit. Since $\widetilde{U}_1$ is transverse to the orbits, this implies that the restriction $\widetilde{f} := \widetilde{f \circ P}_{|\widetilde{U}_1}$ of the submersion $\widetilde{f \circ P}$ still is a submersion. As noted in Remark 2.1.16 this implies the existence of a homomorphism $\Theta_f \colon \Gamma_1 \to \Gamma_2$ satisfying $\widetilde{f} \circ \gamma = \Theta_f(\gamma) \circ \widetilde{f}\ \forall \gamma \in \Gamma_1$. Since $(U_1, \widetilde{U}_1/\Gamma_1, \pi_1)$ is a chart on M/G with the structure given above and we have $\pi_2 \circ \widetilde{f} = f \circ \pi_1$, f is indeed a submersion around x. Since $x \in M/G$ was arbitrary, f is a submersion.

□

Remark 2.2.2.

1. Note that the construction of the injections λ_i above shows that different G-invariant metrics g on M define the same orbifold structure on M/G: If in the part "Chart compatibility" π_1, π_2 are defined with respect to two different G-invariant metrics g_1, g_2, respectively, on M, we can use g_1 to define $\hat{\pi}$, and the maps λ_1, λ_2 will still be injections. However, if we are given a Riemannian manifold (M, g) and a group G of isometries acting almost freely on M, we will usually assume that we use the same metric g for the definition of the charts in the proof of the theorem.

2. It is not hard to see that if N is a manifold, a map $f \colon M/G \to N$ is smooth if and only if $f \circ P \colon M \to N$ is smooth. In fact, the existence of Θ_f as in the end of the proof above is trivial for this special case, since for trivial Γ_2 we can simply choose $\Theta_f \equiv e$.

3. It can be shown that for $\dim M - \dim G = 1$ the manifolds $\psi^{-1}(\tilde{x})$ (which are nothing but images of geodesics in this case) are everywhere perpendicular to the G-orbits (cf. [Mol88] Prop. 3.5), in particular $\hat{\psi}^{-1}(\tilde{y}_1) \subset \psi_1^{-1}(\tilde{x}_1)$ and $g\hat{\psi}^{-1}(\tilde{y}_1) \subset \psi_2^{-1}(\tilde{x}_2)$ in the compatibilty proof above. However, in general the normal distribution $M \ni \tilde{x} \mapsto T_{\tilde{x}}(G\tilde{x})^{\perp} \subset T_{\tilde{x}}M$ (which we will also refer to as the horizontal distribution later on) is not integrable (cf. [Ton97] chapter 5).

4. It can be shown that every orbifold is diffeomorphic to the quotient of a manifold by an almost free Lie group action ([ALR07] Theorem 1.23).

5. If Γ is a discrete (not necessarily finite) group acting properly discontinuously and effectively on a manifold M, the quotient M/Γ also becomes an orbifold, which is by definition a good orbifold ([Thu81], also compare Theorem 2.2.4).

2.2.2 Quotients of Orbifolds by Finite Groups

In this part we will show that a quotient $\mathcal{O}/G$ of an orbifold and a finite group of diffeomorphism on $\mathcal{O}$ carries a canonical orbifold structure. Since this construction gives a typical example for orbifold coverings, we include the following definition, which goes back to Thurston ([Thu81]).

Definition 2.2.3. An *orbifold covering* between two orbifolds $\mathcal{O}_1$, $\mathcal{O}_2$ is a smooth map $P\colon \mathcal{O}_1 \to \mathcal{O}_2$ such that for every $y \in \mathcal{O}_2$ there is an open connected neighbourhood V around y such that $P^{-1}(V)$ is a disjoint union $\bigcup_\alpha U_\alpha$ and such that

- $(V, \tilde{V}/\Gamma, \pi)$ is a chart around y,
- on every U_α there is a chart of the form $(U_\alpha, \tilde{V}/\Gamma_\alpha, \pi_\alpha)$ satisfying $\Gamma_\alpha \subset \Gamma$ and $P \circ \pi_\alpha = \pi$.

Note that an orbifold covering need not be a covering map between the underlying spaces and hence the covering theory on topological spaces does not apply.

Theorem 2.2.4. *Let $\mathcal{O}$ be an orbifold and let G be a finite subgroup of the diffeomorphism group of $\mathcal{O}$. Then there is - up to diffeomorphism - a unique Riemannian orbifold structure on $\mathcal{O}/G$ such that the quotient map $P\colon \mathcal{O} \to \mathcal{O}/G$ is an orbifold covering and such that a map $f\colon \mathcal{O}/G \to \mathcal{N}$ into an arbitrary orbifold $\mathcal{N}$ is a submersion if and only if $f \circ P$ is a submersion.*

Proof. First note that $\mathcal{O}/G$ is second countable because the quotient map $\mathcal{O} \to \mathcal{O}/G$ is open. Let $x \in \mathcal{O}$. Since G is finite and $\mathcal{O}$ is Hausdorff, there is a neighbourhood V of x in $\mathcal{O}$ such that $gV = V \ \forall g \in G_x$ and $gV \cap V = \emptyset \ \forall g \in G \setminus G_x$. Let W be a chart domain around x contained in V and let U be the connected component of $\bigcap_{g \in G_x} gW$ containing x. Then there is a chart $(U, \tilde{U}/\Gamma, \pi)$ around x and

$$gU = U \ \forall g \in G_x,\ gU \cap U = \emptyset \ \forall g \in G \setminus G_x.$$

Moreover, since all elements of G_x are diffeomorphisms, we can, by choosing U sufficiently small, assume that every element of G_x lifts to a diffeomorphism on $\tilde{U}$.

Note that for $x_1, x_2 \in \mathcal{O}$ with $Gx_1 \cap Gx_2 = \emptyset$, we can, since G is finite and $\mathcal{O}$ is Hausdorff, choose V_1, V_2 around x_1, x_2, respectively, as above such that $gV_1 \cap V_2 = \emptyset \ \forall g \in G$. In particular, $\mathcal{O}/G$ is Hausdorff. Note that, with our choice of U, the map $U/G_x \ni [y] \mapsto [y] \in \mathcal{O}/G$ is injective. We now construct a chart over $U/G_x \subset \mathcal{O}/G$: Let $\widetilde{G}_x$ denote the lifts of elements of G_x to $\tilde{U}$, i.e.,

$$\widetilde{G}_x = \{\phi \in \mathrm{Diffeo}(\tilde{U});\ \exists g \in G_x\colon\ \pi \circ \phi = g \circ \pi\}.$$

This is a group containing Γ as a subgroup. Note that the map ψ sending $\phi \in \widetilde{G}_x$ to $g \in G_x$ such that $\pi \circ \phi = g \circ \pi$ is a well-defined homomorphism. ψ is surjective by our choice of U. The kernel of ψ is given by $\{\phi \in \mathrm{Diffeo}(\tilde{U});\ \pi \circ \phi = \pi\}$, which coincides with Γ by Proposition 2.1.3. Hence ψ induces an isomorphism $\widetilde{G}_x/\Gamma \simeq G_x$.

We have the following homeomorphisms: $U/G_x \approx (\tilde{U}/\Gamma)/(\tilde{G}_x/\Gamma) \approx \tilde{U}/\tilde{G}_x$. Hence, with σ denoting the quotient map $U \to U/G_x \subset \mathcal{O}/G$, a chart around $[x] \in \mathcal{O}/G$ is given by $(U/G_x, \tilde{U}/\tilde{G}_x, \sigma \circ \pi)$.

$$
\begin{tikzcd}
\tilde{U} \arrow[d, "\pi"] & & \\
U \arrow[r, "\approx"] \arrow[d, "\sigma"] & \tilde{U}/\Gamma \arrow[d] & \\
U/G_x \arrow[r, "\approx"] & (\tilde{U}/\Gamma)/(\tilde{G}_x/\Gamma) \arrow[r, "\approx"] & \tilde{U}/\tilde{G}_x
\end{tikzcd}
$$

The compatibility of $\mathcal{O}$-charts implies the compatibility of the charts constructed above: If U_1, U_2 are open neighbourhoods of points x_1, x_2, respectively, as above (with corresponding charts $(U_i/G_{x_i}, \tilde{U}_i/\widetilde{G}_{x_i}, \sigma_i \circ \pi_i)$, $i = 1, 2$) and $x \in \mathcal{O}$ such that $[x] \in U_1/G_{x_1} \cap U_2/G_{x_2} \subset \mathcal{O}/G$, then (since $\mathcal{O}$ is an orbifold) there is a chart $(U, \tilde{U}/\Gamma, \pi)$ around x in $\mathcal{O}$ such that $x \in U \subset U_1 \cap U_2$. By choosing U sufficiently small, we can assume that π induces a chart $\sigma \circ \pi$ on $\mathcal{O}/G$ as above. For each $i = 1, 2$ there is an injection λ_i from π to π_i. But $\pi_i \circ \lambda_i = \pi$ implies $\sigma_i \circ \pi_i \circ \lambda_i = \sigma \circ \pi$, i.e., each λ_i is an injection from $\sigma \circ \pi$ to $\sigma_i \circ \pi_i$. Hence, we obtain a smooth orbifold structure on $\mathcal{O}/G$.

To see that P is an orbifold covering, note that with $x \in \mathcal{O}$ and $(U, \tilde{U}/\Gamma, \pi)$ as in the situation above the preimage $P^{-1}(U/G_x)$ is given by the disjoint union $\bigcup_{[g] \in G/G_x} gU$ and over gU an $\mathcal{O}$-chart is given by $(gU, \tilde{U}/\Gamma, g \circ \pi)$.

To see that a map $f\colon \mathcal{O}/G \to \mathcal{N}$ is a submersion if and only if $f \circ P$ is a submersion, first note that every covering is a submersion because in the charts from Definition 2.2.3 lifts of P are given by the identity on $\tilde{V}$. Hence what remains to be shown is that if $f \circ P$ is a submersion then so is f. So let $x \in \mathcal{O}$ and let $(U/G_x, \tilde{U}/\tilde{G}_x, \sigma \circ \pi)$ be a chart as above around $[x] \in \mathcal{O}/G$. Assume that there is a chart $(V, \tilde{V}/\Delta, p)$ around $f(x) \in \mathcal{N}$ and a submersion $\widetilde{f \circ P}\colon \tilde{U} \to \tilde{V}$ satisfying $f \circ P \circ \pi = p \circ \widetilde{f \circ P}$. Then $f \circ \sigma \circ \pi = p \circ \widetilde{f \circ P}$, i.e., $\widetilde{f \circ P}$ also lifts f. The corresponding homomorphism $\tilde{G} \to \Delta$ exists by Remark 2.1.16.

The uniqueness of the orbifold structure on $\mathcal{O}/G$ follows because a covering is a submersion and hence the identity between two different orbifold structures as in the theorem is a diffeomorphism.

□

2.3 Tensor Fields

2.3.1 Tensor Fields on General Orbifolds

Just like an atlas is a collection of germs of charts around every orbifold point, a tensor field can be defined to be a collection of germs of tensor fields (cf. [Sat57]):

Definition 2.3.1. (i) Let $(U_i, \tilde{U}_i/\Gamma_i, \pi_i)$, $i = 1, 2$, be two charts around a point x in an orbifold $\mathcal{O}$ and on each $\tilde{U}_i$ let τ_i be a Γ_i-invariant (r, s)-tensor field. We write $\tau_1 \sim_x \tau_2$ if there is a chart $(U, \tilde{U}/\Gamma, \pi)$ with $x \in U \subset U_1 \cap U_2$ and injections $\lambda_i\colon \tilde{U} \to \tilde{U}_i$ such that $\lambda_1^* \tau_1 = \lambda_2^* \tau_2$.

(ii) A tensor field of type (r,s) over an atlas $\mathfrak{A} = \{(U_\alpha, \tilde{U}_\alpha/\Gamma_\alpha, \pi_\alpha)\}_{\alpha \in I(\mathfrak{A})}$ is a family $\{\tau_\alpha\}_{\alpha \in I(\mathfrak{A})}$ such that each τ_α is a Γ_α-invariant (r,s)-tensor field on $\tilde{U}_\alpha$ and

$$\forall x \in U_\alpha \cap U_\beta \colon \; \tau_\alpha \sim_x \tau_\beta.$$

Note that in (i) we have that $\lambda_1^* \tau_1 = \lambda_2^* \tau_2 \in T^{r,s}(\tilde{U})$ is Γ-invariant: If $\gamma \in \Gamma$ then

$$\gamma^* \lambda_1^* \tau_1 = (\lambda_1 \circ \gamma)^* \tau_1 = (\overline{\lambda}_1(\gamma) \circ \lambda_1)^* \tau_1 = \lambda_1^* \overline{\lambda}_1(\gamma)^* \tau_1 = \lambda_1^* \tau_1.$$

Remark. An orbifold tensor field can also be interpreted as a section in a particular fibre bundle (cf. [BZ07]). However, we prefer the more intuitive notion above.

The following lemma shows that a tensor field over an arbitrary (non-maximal) orbifold atlas can be uniquely extended to a tensor field over a maximal atlas.

Lemma 2.3.2. *Let $\mathfrak{A} = \{(U_\alpha, \tilde{U}_\alpha/\Gamma_\alpha, \pi_\alpha)\}_{\alpha \in I(\mathfrak{A})}$ be an atlas on an orbifold $\mathcal{O}$ and $\{\tau_\alpha\}_{\alpha \in I(\mathfrak{A})}$ an (r,s)-tensor field over $\mathfrak{A}$. If $(U, \tilde{U}/\Gamma, \pi)$ is an orbifold chart which is compatible with $\mathfrak{A}$, then there is a unique Γ-invariant (r,s)-tensor field τ on $\tilde{U}$ such that $\{\tau_\alpha\}_{\alpha \in I(\mathfrak{A})} \cup \{\tau\}$ is an (r,s)-tensor field over the atlas $\mathfrak{A} \cup \{\pi\}$.*

Proof. We will first define tensor fields locally on $\tilde{U}$ and then show that they coincide on the intersections of their respective domains to give a smooth Γ-invariant tensor field τ on $\tilde{U}$ as in the theorem.

The local definition: Let $\tilde{x} \in \tilde{U}$ and let π_α be a chart in $\mathfrak{A}$ around $x := \pi(\tilde{x}) \in U \subset \mathcal{O}$. Since π is compatible with $\mathfrak{A}$, there is a chart $(W, \tilde{W}/G, p)$ (not necessarily in $\mathfrak{A}$) around x with corresponding injections $\lambda \colon \tilde{W} \to \tilde{U}_\alpha$ into π_α and $\phi \colon \tilde{W} \to \tilde{U}$ into π. After composing ϕ with an element of Γ if necessary, we can assume that $\tilde{x} \in \phi(\tilde{W})$.

$$\begin{array}{ccccc} \tilde{U}_\alpha & \xleftarrow{\lambda} & \tilde{W} & \xrightarrow{\phi} & \tilde{U} \\ \downarrow{\scriptstyle \pi_\alpha} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ U_\alpha & \xleftarrow{\supset} & W & \xrightarrow{\subset} & U \end{array}$$

Then $\sigma := \phi_* \lambda^* \tau_\alpha$ is a tensor field on $\phi(\tilde{W})$, which is invariant under $\bar{\phi}(G) \subset \Gamma$ since τ_α is Γ_α-invariant. Next, extend σ to a tensor field on $\Gamma \phi(\tilde{W}) \subset \tilde{U}$ by

$$\sigma_{\gamma \tilde{y}} := \gamma_* \sigma_{\tilde{y}} \text{ for } \gamma \in \Gamma, \tilde{y} \in \phi(\tilde{W})$$

and note that this is well-defined because if $\gamma_1 \tilde{y}_1 = \gamma_2 \tilde{y}_2$ for some $\gamma_i \in \Gamma$, $\tilde{y}_i \in \phi(\tilde{W})$, then $\gamma_2^{-1} \gamma_1 \in \bar{\phi}(G)$ by Proposition 2.1.3 part 1 and hence $\gamma_{1*} \sigma_{\tilde{y}_1} = \gamma_{2*} \sigma_{\tilde{y}_2}$.

The local tensor fields coincide: Now suppose we are given two charts $\pi_i := \pi_{\alpha_i}$, $(W_i, \tilde{W}_i/G_i, p_i)$, $i = 1, 2$, and injections λ_i into π_i and ϕ_i into π as above such that $\Gamma \phi_1(\tilde{W}_1) \cap \Gamma \phi_2(\tilde{W}_2) \neq \emptyset$. We have to show that the induced tensor fields σ_i coincide on $\Gamma \phi_1(\tilde{W}_1) \cap \Gamma \phi_2(\tilde{W}_2) \subset \tilde{U}$. So let $\tilde{y} \in \Gamma \phi_1(\tilde{W}_1) \cap \Gamma \phi_2(\tilde{W}_2)$ be arbitrary and choose $\gamma_i \in \Gamma$, $\tilde{y}_i \in \tilde{W}_i$ such that $\tilde{y} = \gamma_i \phi_i(\tilde{y}_i)$ for $i = 1, 2$.

Now note that $\pi(\tilde{y}) \in U_1 \cap U_2$. Hence, by the definition of an orbifold tensor field there is a chart $(V, \tilde{V}/\Gamma_V, \pi_V)$ around $\pi(\tilde{y})$ with corresponding injections $\psi_i \colon \tilde{V} \to \tilde{U}_i$ into π_i satisfying $\psi_1^*\tau_1 = \psi_2^*\tau_2 \in T^{r,s}(\tilde{V})$. By shrinking V if necessary we can assume that $\pi(\tilde{y}) \in V \subset W_1 \cap W_2$. Moreover, we can assume that $\lambda_i(\tilde{y}_i) \in \psi_i(\tilde{V}) \subset \lambda_i(\tilde{W}_i)$ (by composing ψ_i with an element of Γ_i if necessary and noting that τ_i is Γ_i-invariant).

Denote by $\lambda_i^{-1} \colon \lambda_i(\tilde{W}_i) \to \tilde{W}_i$ the (left) inverse of λ_i. Then each $j_i := \lambda_i^{-1} \circ \psi_i$ is an injection from π_V into p_i and $\tilde{y}_i \in j_i(\tilde{V})$. We obtain the following diagram. (Note that it is not necessarily commutative in the triangle in the centre formed by j_i and ϕ_i.)

$$\begin{tikzcd}
 & & \tilde{V} \arrow[dll, "\psi_1"'] \arrow[dl, "j_1"] \arrow[dr, "j_2"'] \arrow[drr, "\psi_2"] & & \\
\tilde{U}_1 \arrow[d, "\pi_1"'] & \tilde{W}_1 \arrow[l, "\lambda_1"] \arrow[r, "\phi_1"'] \arrow[d, "p_1"'] & \tilde{U} \arrow[d, "\pi"'] & \tilde{W}_2 \arrow[l, "\phi_2"] \arrow[r, "\lambda_2"'] \arrow[d, "p_2"] & \tilde{U}_2 \arrow[d, "\pi_2"] \\
U_1 & W_1 \arrow[l, "\supset"'] \arrow[r, "\subset"] & U & W_2 \arrow[l, "\supset"'] \arrow[r, "\subset"] & U_2
\end{tikzcd}$$

Since both $\gamma_i \circ \phi_i \circ j_i$ are injections from π_V into π, there is $\gamma \in \Gamma$ such that $\gamma_2 \circ \phi_2 \circ j_2 = \gamma\gamma_1 \circ \phi_1 \circ j_1$ (Proposition 2.1.3). But this relation implies that

$$\gamma^{-1}\tilde{y} = \gamma^{-1}\gamma_2 \circ \phi_2(\tilde{y}_2) \subset \gamma^{-1}\gamma_2 \circ \phi_2 \circ j_2(\tilde{V}) = \gamma_1\phi_1 j_1(\tilde{V}).$$

Since also $\tilde{y} \in \gamma_1\phi_1 j_1(\tilde{V})$, we have $\gamma \in \overline{\gamma_1 \circ \phi_1 \circ j_1}(\Gamma_V)$ by Proposition 2.1.3 part 1.

Set $\tau_V := \psi_1^*\tau_1 = \psi_2^*\tau_2 \in T^{r,s}(\tilde{V})$. Then on $\gamma_2 \circ \phi_2 \circ j_2(\tilde{V})$ we have

$$\begin{aligned}
\sigma_2 &= \gamma_{2*}\phi_{2*}\lambda_2^*(\tau_{2|\psi_2(V)}) = (\gamma_2 \circ \phi_2 \circ j_2)_*\tau_V = (\gamma\gamma_1 \circ \phi_1 \circ j_1)_*\tau_V \\
&= \gamma_*(\gamma_1 \circ \phi_1 \circ j_1)_*\tau_V = \gamma_{1*}\phi_{1*}j_{1*}\tau_V = \gamma_{1*}\phi_{1*}\lambda_1^*(\tau_{1|\psi_1(V)}) \\
&= \sigma_1,
\end{aligned}$$

where we used that $(\gamma_1 \circ \phi_1 \circ j_1)_*\tau_V$ is $\overline{\gamma_1\phi_1 j_1}(\Gamma_V)$-invariant because τ_V is Γ_V-invariant.

In particular $\sigma_1 = \sigma_2$ in $\tilde{y}$, hence the tensor fields σ around $\tilde{x}$ with $\tilde{x}$ running over $\tilde{U}$ patch together to give a well-defined tensor field τ on $\tilde{U}$, which is Γ-invariant by construction.

All in all, this gives an orbifold tensor field $\{\tau_\alpha\}_\alpha \cup \{\tau\}$ over $\mathfrak{A} \cup \{\pi\}$. The uniqueness of τ is clear. □

Corollary 2.3.3. *If $\{\tau_\alpha\}_\alpha$ is a tensor field over an orbifold atlas $\{(U_\alpha, \tilde{U}_\alpha/\Gamma_\alpha, \pi_\alpha)\}_{\alpha \in I(\mathfrak{A})}$ on a manifold M, it defines a unique (manifold) tensor field τ on M.*

Proof. Just apply the lemma above to the chart $(M, M/\{\mathrm{id}_M\}, \mathrm{id}_M)$. □

Of course, if all Γ_α in the corollary above are trivial, the tensor field τ is just given by pushing forward each τ_α via the diffeomorphism π_α and patching the local tensor fields on M together.

Definition 2.3.4. A *tensor field over an orbifold* $\mathcal{O} = (X, \mathfrak{A})$ is a tensor field over the maximal atlas $\mathfrak{A}$. The set of all tensor fields over the orbifold $\mathcal{O}$ will be denoted by $T^{r,s}(\mathcal{O})$.

Given a tensor field τ over an atlas contained in the maximal atlas $\mathfrak{A}$, its unique extension to $\mathfrak{A}$ (which exists by Lemma 2.3.2) will be called the tensor field on $\mathcal{O}$ *induced* by τ.

Remark 2.3.5. Given a tensor field $\tau = \{\tau_\alpha\}_{\alpha \in I(\mathfrak{A})}$ on an orbifold $(X, \mathfrak{A})$ with $\mathfrak{A} = \{(U_\alpha, \tilde{U}_\alpha/\Gamma_\alpha, \pi_\alpha)\}_{\alpha \in I(\mathfrak{A})}$ and a chart $(U, \tilde{U}/\Gamma, \pi) \in \mathfrak{A}$ we will also write $\tau_\pi := \tau_\alpha$ if $\pi = \pi_\alpha$. Moreover, we write τ_{reg} (or τ^{reg}) for $\tau_{\pi^{\mathrm{reg}}} \in T^{r,s}(\mathcal{O}^{\mathrm{reg}})$ with π^{reg} the chart on $\mathcal{O}$ given by $(\mathcal{O}^{\mathrm{reg}}, \mathcal{O}^{\mathrm{reg}}/\{\mathrm{id}\}, \mathrm{id})$.

Note that we have a canonical identification of $T^{0,0}(\mathcal{O})$ with the space $C^\infty(\mathcal{O})$ of real-valued smooth functions on $\mathcal{O}$. Moreover, note that $T^{r,s}(\mathcal{O})$ is naturally a $C^\infty(\mathcal{O})$-module: Given two tensor fields over $\mathcal{O}$, we simply define the sum locally - and similarly for the product of a smooth function with a tensor field. It is easy to show that the resulting tensor fields again satisfy the conditions of Definition 2.3.1.

It is also not hard to see that every (r, s)-tensor field on $\mathcal{O}$ canonically gives a $C^\infty(\mathcal{O})$-multilinear map $\Omega^1(\mathcal{O})^r \times \mathcal{V}(\mathcal{O})^s \to C^\infty(\mathcal{O})$ in the following way: Given $\{\tau_\alpha\}_\alpha \in T^{r,s}(\mathcal{O})$, 1-forms $\{\omega^i_\alpha\}_\alpha, i = 1, \dots, r$, and vector fields $X^j = \{X^j_\alpha\}_\alpha, j = 1, \dots, s$, define functions $f_\alpha \in C^\infty(\tilde{U}_\alpha)^{\Gamma_\alpha}$ by

$$f_\alpha := \tau_\alpha(\omega^1_\alpha, \dots, \omega^r_\alpha, X^1_\alpha, \dots, X^s_\alpha).$$

Then $\{f_\alpha\}_\alpha \in T^{0,0}(\mathcal{O})$, hence it gives a function $f \in C^\infty(\mathcal{O})$. The assignment

$$(\omega_1, \dots, \omega_r, X_1, \dots, X_s) \mapsto f$$

is $C^\infty(\mathcal{O})$-multilinear because the τ_α are $C^\infty(\tilde{U}_\alpha)^{\Gamma_\alpha}$-multilinear.

A symmetric positive definite $(0, 2)$-tensor field is called a Riemannian orbifold metric. Moreover, we write $\mathcal{V}(\mathcal{O}) := T^{1,0}(\mathcal{O})$ for the $C^\infty(\mathcal{O})$-module of vector fields on $\mathcal{O}$. For a smooth function $f \in C^\infty(\mathcal{O})$ on a Riemannian orbifold $(\mathcal{O}, g)$ (where g is a Riemannian orbifold metric) we use the gradients of local lifts $f \circ \pi$ to define the orbifold vector field $\operatorname{grad} f \in \mathcal{V}(\mathcal{O})$.

Given a smooth function $f\colon\ \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds, the pullbacks via local lifts $\tilde{f}$ can be used to define a *pullback* operator $f^*\colon\ T^{0,k}(\mathcal{O}_2) \to T^{0,k}(\mathcal{O}_1)$ which we will define soon (see Lemma 2.3.7) but first we need the following technical lemma.

Lemma 2.3.6. *Let $(U, \tilde{U}/\Gamma, \pi)$ be an orbifold chart, $x \in U$, $\tilde{x} \in \pi^{-1}(x)$ and let $\tilde{V} \subset \tilde{U}$ be a neighbourhood of $\tilde{x}$. Then there is a neighbourhood $U' \subset U$ of x and a chart $(U', \tilde{U}'/\Gamma', \pi')$ with an injection λ into π such that $\tilde{x} \in \lambda(\tilde{U}') \subset \tilde{V}$ and $\Gamma' \simeq \mathrm{Iso}(x)$.*

Proof. Since Γ is finite, there is a neighbourhood $\tilde{W} \subset \tilde{V}$ of $\tilde{x}$ such that $\gamma\tilde{W} \cap \tilde{W} = \emptyset\ \forall \gamma \in \Gamma \setminus \Gamma_{\tilde{x}}$. Then let $\tilde{U}'$ be the connected component of $\bigcap_{\gamma \in \Gamma_{\tilde{x}}} \gamma\tilde{W}$ containing $\tilde{x}$, $\Gamma' := \Gamma_{\tilde{x}} \subset \Gamma$, $U' := \pi(\tilde{U}')$ and $\pi' := \pi_{|\tilde{U}'}$. This gives the desired chart, where we choose λ to be the inclusion $\tilde{U}' \to \tilde{U}$. □

Lemma 2.3.7. *Let $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ be a smooth map and let $\tau \in T^{0,k}(\mathcal{O}_2)$ be a covariant tensor field on $\mathcal{O}_2$. For $x \in \mathcal{O}_1$ let $(U_i, \tilde{U}_i/\Gamma_i, \pi_i)$, $i = 1, 2$, be charts over x and $f(x)$, respectively, and $\tilde{f} \in C^\infty(\tilde{U}_1, \tilde{U}_2), \Theta\colon \Gamma_1 \to \Gamma_2$ as in Definition 2.1.15.*

Let $\tilde{\tau} := \tau_{\pi_2} \in T^{0,k}(\tilde{U}_2)^{\Gamma_2}$ be the tensor field on $\tilde{U}_2$ given by $\tau \in T^{0,k}(\mathcal{O}_2)$ and set $\sigma^x := \tilde{f}^*\tilde{\tau} \in T^{0,k}(\tilde{U}_1)$. Then:

The set $\sigma := \{\sigma^x\}_{x\in\mathcal{O}_1}$ defined above gives a tensor field on the corresponding atlas of $\mathcal{O}_1$.

Proof. For a fixed $x \in \mathcal{O}_1$ the tensor field $\tilde{\sigma} := \sigma^x$ is obviously Γ_1-invariant: For $\gamma \in \Gamma_1$ we have

$$\gamma^*\tilde{\sigma} = \gamma^*\tilde{f}^*\tilde{\tau} = \tilde{f}^*\Theta(\gamma)^*\tilde{\tau} = \tilde{f}^*\tilde{\tau} = \tilde{\sigma}.$$

To show compatibility of two local tensor fields in the sense of Definition 2.3.1 assume that we are given $x, x' \in \mathcal{O}_1$ with local lifts $\tilde{f}$, $\tilde{f}'$ of f, corresponding charts π_i, π_i', $i = 1, 2$, and pullbacks $\tilde{\sigma} = \tilde{f}^*\tilde{\tau}$, $\tilde{\sigma}' = \tilde{f}'^*\tilde{\tau}'$ with $\tilde{\tau}$ and $\tilde{\tau}'$ given by τ and such that $U_1 \cap U_1' \neq \emptyset$.

Let $y \in U_1 \cap U_1'$. Since τ is a tensor field, there is a chart $(V, \tilde{V}/G, p)$ around $f(y)$ in $\mathcal{O}_2$ with $V \subset U_2 \cap U_2'$ and injections λ, λ' from p into π_2, π_2' such that

$$\tau_p = \lambda^*\tilde{\tau} = \lambda'^*\tilde{\tau}' \in T^{0,k}(\tilde{V})^G.$$

Then choose $\tilde{y} \in \pi_1^{-1}(y) \subset \tilde{U}_1$ and $\tilde{y}' \in \pi_1'^{-1}(y) \subset \tilde{U}_1'$.

By composing λ with an appropriate element of Γ_2 and λ' with an appropriate element of Γ_2', we can assume that $\tilde{f}(\tilde{y}) \in \lambda(\tilde{V})$ and $\tilde{f}'(\tilde{y}') \in \lambda'(\tilde{V})$. Moreover, by Lemma 2.3.6, we can assume that $\tilde{f}(\tilde{U}_1) \subset \lambda(\tilde{V})$, $\tilde{f}'(\tilde{U}_1') \subset \lambda'(\tilde{V})$.

Since π_1 and π_1' are compatible, there is a chart $(W, \widetilde{W}/\Gamma, \pi)$ around y with injections μ, μ' into π_1 and π_1' and we have the following commutative diagram:

$$\begin{tikzcd}
\tilde{W} \arrow[r, "\mu"] \arrow[d, "\pi"'] & \tilde{U}_1 \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"'] & \tilde{U}_2 \arrow[d, "\pi_2"'] & \tilde{V} \arrow[l, "\lambda"'] \arrow[r, "\lambda'"] \arrow[d, "p"'] & \tilde{U}_2' \arrow[d, "\pi_2'"] & \tilde{U}_1' \arrow[l, "\tilde{f}'"'] \arrow[d, "\pi_1'"] & \tilde{W} \arrow[l, "\mu'"'] \arrow[d, "\pi"] \\
W \arrow[r, "\subset"] & U_1 \arrow[r, "f"] & U_2 & V \arrow[l, "\supset"'] \arrow[r, "\subset"] & U_2' & U_1' \arrow[l, "f"'] & W \arrow[l, "\supset"']
\end{tikzcd}$$

Let $w \in \pi^{-1}(y)$. By composing μ and μ' with appropriate elements of Γ_1 and Γ_1', respectively, we can assume that $\mu(w) = \tilde{y}$ and $\mu'(w) = \tilde{y}'$.

Since the diagram above commutes, we have

$$p \circ \lambda^{-1} \circ \tilde{f} \circ \mu = f \circ \pi = p \circ \lambda'^{-1} \circ \tilde{f}' \circ \mu'. \tag{2.2}$$

Now for $g \in G$ set

$$\widetilde{W}_g := \{v \in \widetilde{W};\ g \circ \lambda^{-1} \circ \tilde{f} \circ \mu(v) = \lambda'^{-1} \circ \tilde{f}' \circ \mu'(v)\}$$

and note that (2.2) implies $\widetilde{W} = \bigcup_{g\in G} \widetilde{W}_g$. On each $\mathring{\widetilde{W}}_g$ we have

$$\begin{aligned}\mu^*\tilde{\sigma} &= \mu^*\tilde{f}^*\tilde{\tau} = \mu^*\tilde{f}^*(\lambda^{-1})^*\tau_p = \mu^*\tilde{f}^*(\lambda^{-1})^* g^*\tau_p \\ &= (g\circ\lambda^{-1}\circ\tilde{f}\circ\mu)^*\tau_p = (\lambda'^{-1}\circ\tilde{f}'\circ\mu')^*\tau_p \\ &= \mu'^*\tilde{f}'^*\tilde{\tau}' = \mu'^*\tilde{\sigma}'.\end{aligned}$$

To see that $\mu^*\tilde{\sigma} = \mu'^*\tilde{\sigma}'$ on all of $\widetilde{W}$, we will show that $\bigcup_{g\in G} \mathring{\widetilde{W}}_g$ is dense in $\widetilde{W}$. To this end let $v \in \widetilde{W}$ and let $B \subset \widetilde{W}$ be an arbitrary open neighbourhood of v in $\widetilde{W}$. Then we have the decomposition $B = \bigcup_{g\in G}(B \cap \widetilde{W}_g)$. Since $B \cap \widetilde{W}_g$ is closed in B and G is finite, Baire's Theorem (for locally compact Hausdorff spaces) implies that there is $g \in G$ such that $B \cap \mathring{\widetilde{W}}_g = (B \cap \widetilde{W}_g)^\circ$ is non-empty and hence $B \cap \bigcup_{g\in G} \mathring{\widetilde{W}}_g$ is non-empty. Since $v \in \widetilde{W}$ was arbitrary, $\bigcup_{g\in G} \mathring{\widetilde{W}}_g$ is dense in $\widetilde{W}$ and therefore $\mu^*\tilde{\sigma} = \mu'^*\tilde{\sigma}'$ on $\widetilde{W}$. Hence, $\tilde{\sigma}$ and $\tilde{\sigma}'$ satisfy the conditions from Definition 2.3.1. All in all, σ is a smooth tensor field on the corresponding atlas on $\mathcal{O}_1$. □

Definition 2.3.8. The $(0,k)$-tensor field on $\mathcal{O}_1$ induced by σ from the lemma above is called the pullback of τ by f and denoted by $f^*\tau$.

If $\omega \in \Omega^k(\mathcal{O})$ is a k-form on $\mathcal{O}$ (i.e. $\omega = \{\omega_\alpha\}_\alpha \in T^{0,k}(\mathcal{O})$ such that each ω_α is alternating), we can use the local d-operators to define a $(k+1)$-form $d\omega$ on $\mathcal{O}$, and the relation $d\circ f^* = f^*\circ d$ (for f a smooth map) carries over to orbifolds.

2.3.2 Tensor Fields on Quotient Orbifolds

In this section we come back to the two settings of Section 2.2. First we consider the situation of Theorem 2.2.1 and observe which tensor fields on M correspond to tensor fields on M/G. If G is a Lie group acting smoothly and effectively on a manifold M, a covariant tensor field $\tau \in T^{0,k}(M)$ on M is called *G-horizontal* if for every $\tilde{x} \in M$ and $X_1, \ldots, X_k \in T_{\tilde{x}}M$ we have $\tau(X_1, \ldots, X_k) = 0$ as soon as at least one vector $X_i \in T_{\tilde{x}}M$ is vertical, i.e. $X_i \in T_{\tilde{x}}(G\tilde{x}) \subset T_{\tilde{x}}M$.

From now on we assume moreover that G is compact and connected and acts almost freely (i.e. with finite stabilizers) on M. Write

$$\overline{T^{0,k}}^G(M) := \{\tau \in T^{0,k}(M);\ \tau \text{ is } G\text{-horizontal and } G\text{-invariant}\}$$

and note that this is a module over the ring $C^\infty(M)^G$ of G-invariant functions on M (which is canonically isomorphic to $C^\infty(M/G)$ by Remark 2.2.2 part 2). With the latter identification we have:

Theorem 2.3.9. *Let G be a compact Lie group acting effectively and almost freely on the manifold M, denote by M/G the smooth orbifold from Theorem 2.2.1, write $P\colon M \to M/G$ for the quotient map and $P^*\colon T^{0,k}(M/G) \to T^{0,k}(M)$ for the pull-back operator from Definition 2.3.8. Then the corestriction*

$$P^*\colon T^{0,k}(M/G) \to \overline{T^{0,k}}^G(M)$$

is a $C^\infty(M/G)$-isomorphism.

Proof. Linearity over $\mathbb{R}$ and even over $C^\infty(M/G)$ is obvious.

We first give the inverse of P^*: Let $\tau \in \overline{T^{0,k}}^G(M)$. Fix a G-invariant metric on M and equip M/G with the corresponding atlas given in the proof of Theorem 2.2.1. For $\tilde{x} \in M$ consider a chart $(U, \tilde{U}/\Gamma, \pi)$ given by $U = B^\varepsilon(G\tilde{x})/G$, $\tilde{U} = \psi^{-1}(\tilde{x})$, $\Gamma = G_{\tilde{x}}$ and $\pi = P_{|\tilde{U}}$. We write $\iota_S\colon S \to M$ for the inclusion of a subset S of M and set $\tau^{\tilde{x}} := \iota_{\tilde{U}}^* \tau$. Since τ is G-invariant, $\tau^{\tilde{x}} \in T^{k,0}(\tilde{U})$ is Γ-invariant. To see that $\{\tau^{\tilde{x}}\}_{\tilde{x}\in M}$ gives a smooth tensor field, consider $\tilde{x}_1, \tilde{x}_2 \in M$ with corresponding charts π_1, π_2 and local tensor fields $\tau_i := \tau^{\tilde{x}_i} = \tau_{|\psi_i^{-1}(\tilde{x}_i)}$, $i = 1, 2$, such that $B^{\varepsilon_1}(G\tilde{x}_1)/G \cap B^{\varepsilon_2}(G\tilde{x}_2)/G \neq \emptyset$. Let $y \in B^{\varepsilon_1}(G\tilde{x}_1)/G \cap B^{\varepsilon_2}(G\tilde{x}_2)/G$ and choose $\tilde{y}_i \in \pi_i^{-1}(y)$, $\hat{\pi}$, h_i, $\hat{\psi}$, λ_i as in the proof of Theorem 2.2.1. Moreover, set $\hat{\tau} := \iota^*_{\hat{\psi}^{-1}(\tilde{y}_1)} \tau$.

We want to show that $\lambda_1^* \tau_1 = \hat{\tau} = \lambda_2^* \tau_2$. To this end let $\tilde{r} \in \hat{\psi}^{-1}(\tilde{y}_1)$ and $X_1, \ldots, X_k \in T_{\tilde{r}}(\hat{\psi}^{-1}(\tilde{y}_1)) \subset T_{\tilde{r}} M$. Then for $i = 1, 2$ there is $g_i \in G$ such that $g_i \tilde{r} = \lambda_i(\tilde{r})$. Set

$$\bar{g}_i := g_{i|\hat{\psi}^{-1}(\tilde{y}_1)}\colon \hat{\psi}^{-1}(\tilde{y}_1) \to g_i(\hat{\psi}^{-1}(\tilde{y}_1)).$$

Since τ is horizontal and G-invariant, we indeed have

$$\begin{aligned}
\lambda_i^* \tau_i(X_1, \ldots, X_k) &= \tau_i(\lambda_{i*} X_1, \ldots, \lambda_{i*} X_k) \\
&= \tau((\lambda_i \circ \bar{g}_i^{-1})_* \bar{g}_{i*} X_1, \ldots, (\lambda_i \circ \bar{g}_i^{-1})_* \bar{g}_{i*} X_k) \\
&= \tau(\bar{g}_{i*} X_1, \ldots, \bar{g}_{i*} X_k) = \tau(X_1, \ldots, X_k) \\
&= \hat{\tau}(X_1, \ldots, X_k),
\end{aligned}$$

where for the third "=" we used that for $j = 1, \ldots, k$ the difference of $v_j := (\lambda_i \circ \bar{g}_i^{-1})_* \bar{g}_{i*} X_j$ and $w_j := \bar{g}_{i*} X_j$ in $T_{\lambda_i(\tilde{r})} M$ is vertical (because $P \circ \lambda_i \circ \bar{g}_i^{-1} = P$ on $g_i(\hat{\psi}^{-1}(\tilde{y}_1))$) and applied the identity

$$v_1 \otimes \cdots \otimes v_k - w_1 \otimes \cdots \otimes w_k = \sum_{j=1}^k w_1 \otimes \cdots \otimes w_{j-1} \otimes (v_j - w_j) \otimes v_{j+1} \otimes \cdots \otimes v_k$$

together with the horizontality of τ.

Since the $\tilde{x}_i$ and y were chosen arbitrarily, we conclude that $\{\tau^{\tilde{x}}\}_{\tilde{x}\in M}$ induces a $(0,k)$-tensor field on M/G, which we will denote by $\Psi(\tau)$.

To show that the map $\Psi\colon \overline{T^{0,k}}^G(M) \to T^{0,k}(M/G)$ constructed above is the inverse to P^*, first recall from the proof of Theorem 2.2.1 that for every $\tilde{x} \in M$ we locally have the commutative diagram (see (2.1) for the exact definition of ϕ)

$$\begin{tikzcd}
 & \psi^{-1}(V) \arrow[r, "\phi"] \arrow[d, "\mathrm{id}"] & \psi^{-1}(\tilde{x}) \arrow[d, "\pi"] & \\
M \supset & \psi^{-1}(V) \arrow[r, "P"] & B^\varepsilon(G\tilde{x})/G & \subset M/G
\end{tikzcd}$$

with $\phi_{|\psi^{-1}(\tilde{x})}$ the identity on $\psi^{-1}(\tilde{x})$ and ϕ a G-invariant submersion.

Now to see that Ψ is a right-inverse to P^*, start with $\tau \in \overline{T^{0,k}}^G(M)$ and apply the construction above. Let $\tilde{x} \in M$, choose ε and V such that there is ϕ as above and let $X \in T_{\tilde{x}}M$. We can write $X = Y + Z$ with $Y \in T_{\tilde{x}}(\psi^{-1}(\tilde{x}))$ horizontal and $Z \in T_{\tilde{x}}(G\tilde{x})$ vertical. Since $\phi_{|\psi^{-1}(\tilde{x})} = \mathrm{id}_{\psi^{-1}(\tilde{x})}$, we have

$$(\phi^*\tau^{\tilde{x}})(Y) = \tau^{\tilde{x}}(Y) = \tau(Y).$$

Moreover, since $\phi_* Z = 0$ and τ is horizontal, we also have $\phi^*\tau^{\tilde{x}}(Z) = 0 = \tau(Z)$ and therefore $\phi^*\tau^{\tilde{x}}(X) = \tau(X)$. Since $\tilde{x} \in M$ was arbitrary, we have shown that τ is the pullback by P of the tensor field on M/G induced by $\{\tau^{\tilde{x}}\}_{\tilde{x}\in M}$. In other words, $P^* \circ \Psi(\tau) = \tau$.

In particular, P^* is surjective. It remains to show injectivity of P^*. But if $P^*\sigma = 0$ for some $\sigma \in T^{0,k}(M/G)$ then, in the diagram above, $\phi^*\tilde{\sigma}$ for the corresponding $\tilde{\sigma} \in T^{0,k}(\psi^{-1}(\tilde{x}))$. This implies $\tilde{\sigma} = 0$ because ϕ is a submersion of manifolds. Since $\tilde{x}$ was arbitrary, we have $\sigma = 0$. $\square$

Definition 2.3.10. A *Riemannian submersion* between two Riemannian orbifolds $(\mathcal{O}_1, g_1)$, $(\mathcal{O}_2, g_2)$ is a smooth map $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ such that for every $x \in \mathcal{O}_1$ there are charts $(U_1, \tilde{U}_1/\Gamma_1, \pi_1)$ around x and $(U_2, \tilde{U}_2/\Gamma_2, \pi_2)$ around $f(x)$ and a Riemannian submersion $\tilde{f}\colon (\tilde{U}_1, g_{1_{\pi_1}}) \to (\tilde{U}_2, g_{2_{g_2}})$ such that $f \circ \pi_1 = \pi_2 \circ \tilde{f}$.

Remark. We had already noted in Remark 2.1.16 that in the situation of the definition above there is a unique homomorphism $\Theta : \Gamma_1 \to \Gamma_2$ for which $\tilde{f} \circ \gamma = \Theta(\gamma) \circ \tilde{f}$ $\forall \gamma \in \Gamma_1$. In particular, a Riemannian submersion is of course a submersion.

It is easy to see that on Riemannian orbifolds all injections are local isometries and it easily follows that if $f\colon (\mathcal{O}_1, g_1) \to (\mathcal{O}_2, g_2)$ is a Riemannian orbifold submersion then *every* lift $\tilde{f}$ as in Definition 2.1.15 is a Riemannian manifold submersion.

Corollary 2.3.11. *Let G be a compact Lie group acting isometrically and almost freely on a Riemannian manifold (M, g). Then there is a unique Riemannian orbifold metric g^G on M/G such that $P\colon (M, g) \to (M/G, g^G)$ is a Riemannian orbifold submersion.*

Proof. Let g^G be the preimage of $g(\mathrm{pr}_h \cdot, \mathrm{pr}_h \cdot) \in \overline{T^{0,2}}^G(M)$ under the isomorphism P^* from Theorem 2.3.9 (where $\mathrm{pr}_h(X) := X^h$ denotes the orthogonal projection onto the horizontal space). Then $P\colon (M, g) \to (M/G, g^G)$ is a Riemannian submersion: For each $\tilde{x} \in M$ a lift of the quotient map $P\colon M \to M/G$ around $\tilde{x}$ is given by the map

$$\phi = \left(h_{|\psi^{-1}(\tilde{x})}\right)^{-1} \circ h\colon \psi^{-1}(V) \to \psi^{-1}(\tilde{x})$$

from (2.1) with $h\colon \psi^{-1}(V) \to S$ the submersion in the proof of Theorem 2.2.1. Since the elements of G induce isometries between the horizontal spaces in M, we can define a Riemannian metric g^S on S such that $h\colon (\psi^{-1}(V), g) \to (S, g^S)$ becomes a Riemannian submersion. Then

$$\left(h_{|\psi^{-1}(\tilde{x})}\right)^{-1} : (S, g^S) \to (\psi^{-1}(\tilde{x}), g(\mathrm{pr}_h \cdot, \mathrm{pr}_h \cdot))$$

is an isometry and hence

$$\phi\colon\ (\psi^{-1}(V), g) \to (\psi^{-1}(\tilde{x}), g(\mathrm{pr}_h \cdot, \mathrm{pr}_h \cdot))$$

is a Riemannian submersion. Since we can do this for all $\tilde{x} \in M$, the map $P\colon\ (M, g) \to (M/G, g^G)$ is indeed a Riemannian orbifold submersion.

For uniqueness note that if $P\colon\ (M, g) \to (M/G, g')$ is a Riemannian submersion, then for every $\tilde{x} \in M$ the above lift ϕ around $\tilde{x}$ has to be a Riemannian submersion. This uniquely defines the metric g'_π on $\psi^{-1}(\tilde{x})$ (where $\pi = P_{|\psi^{-1}(\tilde{x})}$ is the corresponding chart around $P(\tilde{x}) \in M/G$). Hence g' is uniquely determined on M/G. □

Theorem 2.3.12. *Let (M, g) and G be as in Corollary 2.3.11 and let F be a G-equivariant isometry on (M, g). Then the homeomorphism f on M/G induced by F is an isometry with respect to the submersion metric g^G.*

Proof. As usual, we denote the quotient map $M \to M/G$ by P. To see that f given by $f(P(\tilde{x})) := P(F(\tilde{x}))$ is smooth let $\tilde{x} \in M$ and choose ε, V, ψ as in the proof of Theorem 2.2.1. This gives a chart $(B^\varepsilon(G\tilde{x})/G, \psi^{-1}(\tilde{x})/G_{\tilde{x}}, P_{|\psi^{-1}(\tilde{x})})$ around $x := P(\tilde{x}) \in M/G$. Setting

$$\psi' := F \circ \psi \circ (F_{|\psi^{-1}(V)})^{-1}\colon\ F(\psi^{-1}(V)) \to F(V),$$

we obtain a chart $(B^\varepsilon(GF(\tilde{x}))/G, \psi'^{-1}(F(\tilde{x}))/G_{F(\tilde{x})}, P_{|\psi'^{-1}(F(\tilde{x}))})$ on M/G around $P(F(\tilde{x})) = f(x)$.

With respect to these two charts a lift of f is given by $\tilde{f} := F_{|\psi^{-1}(\tilde{x})}\colon\ \psi^{-1}(\tilde{x}) \to \psi'^{-1}(F(\tilde{x}))$ with corresponding homeomorphism $\Theta_f\colon\ G_{\tilde{x}} \ni g \mapsto F \circ g \circ F^{-1} \in G_{F(\tilde{x})}$. But this is an isometry with respect to g^G, since $F_{|\psi^{-1}}(V)$ is a G-equivariant isometry and hence its differential maps vertical to vertical and horizontal (with respect to g) to horizontal vectors. □

Generalizing the respective notion on manifolds we make the following definition.

Definition 2.3.13. An action of a Lie group G on an orbifold is called smooth if the corresponding map $G \times \mathcal{O} \to \mathcal{O}$ is smooth (cf. Definition 2.1.21 for the definition of the smooth structure on $G \times \mathcal{O}$).

We then have the following corollary of the theorem above.

Corollary 2.3.14. *Let (M, g) and G be as in Theorem 2.3.12 and let H be a Lie group acting smoothly and isometrically on (M, g) such that the H- and the G-action commute. Then the H-action on M canonically induces a smooth and isometric H-action on M/G.*

Proof. By Theorem 2.3.12 the given H-action on M induces an H-action on $(M/G, g^G)$ by isometries. It remains to show that this action is smooth. So let $\phi\colon\ H \times M \to M$, $\overline{\phi}\colon\ H \times M/G \to M/G$ denote the two given H-actions (with ϕ smooth) and let $P\colon\ M \to M/G$ denote the quotient map.

Now note that ϕ is a submersion since $d\phi_{(h,x)}(0,X) = dh_x X$ for $(h,x) \in H \times M, X \in T_x M$. Since P also is a submersion, the relation

$$\overline{\phi} \circ (\mathrm{id}_H, P) = P \circ \phi$$

and Theorem 2.2.1 (applied to the quotient map (id_H, P)) imply that $\overline{\phi}$ is a submersion; in particular, it is smooth. □

We now consider an analogous construction for vector fields. Choose a G-invariant metric g on M and set

$$\overline{\mathcal{V}}_g^G(M) := \{X \in \mathcal{V}(M);\ X \text{ is } G\text{-horizontal and } G\text{-invariant}\},$$

where X is called G-horizontal if $X_x \in T_x(Gx)^\perp$ with respect to g for all $x \in M$. Moreover, we denote by $M_G \subset M$ the open set of all elements of M which are not fixed by any nontrivial element of G. In other words, $M_G = \{\tilde{x} \in M;\ G_{\tilde{x}} = \{e\}\}$. By the last statement of Theorem 2.2.1, M_G is precisely the preimage of $(M/G)^{\mathrm{reg}}$ under P.

Theorem 2.3.15. *Given a compact connected Lie group G acting isometrically, effectively and almost freely on a Riemannian manifold (M,g), there is a unique $C^\infty(M/G)$-linear isomorphism $\Phi\colon \overline{\mathcal{V}}_g^G(M) \to \mathcal{V}(M/G)$ which satisfies $(\Phi(X))^{reg}_{P(\tilde{x})} = P_*(X_{\tilde{x}})$ for all $\tilde{x} \in M_G$ and $X \in \overline{\mathcal{V}}_g^G(M)$, where P_* denotes the differential of $P_{|M_G}\colon M_G \to (M/G)^{reg}$.*

Proof. Uniqueness follows from the fact that $(M/G)^{\mathrm{reg}}$ is dense in M/G. To show existence we proceed in three steps.

Definition of Φ: First, let $X \in \overline{\mathcal{V}}_g^G(M)$ and $x \in M/G$. Choose $\tilde{x} \in P^{-1}(x)$ and let $(U, \tilde{U}/\Gamma, \pi) = (B^\varepsilon(G\tilde{x})/G, \psi^{-1}(\tilde{x})/G_{\tilde{x}}, P_{|\psi^{-1}(\tilde{x})})$ be a chart around x as in the proof of Theorem 2.2.1. For $\tilde{y} \in \tilde{U} \subset M$ let $\mathrm{pr}^G_{T_{\tilde{y}}\tilde{U}}$ denote the projection from $T_{\tilde{y}}M$ onto $T_{\tilde{y}}\tilde{U}$ with kernel $T_{\tilde{y}}(G\tilde{y})$ and note that

$$\forall \tilde{y} \in \tilde{U}, \gamma \in \Gamma\colon\ \gamma_* \circ \mathrm{pr}^G_{T_{\tilde{y}}\tilde{U}} = \mathrm{pr}^G_{T_{\gamma\tilde{y}}\tilde{U}} \circ \gamma_*\colon\ T_{\tilde{y}}M \to T_{\gamma\tilde{y}}\tilde{U}$$

(as is easily seen by applying both sides to a vector in $T_{\tilde{y}}(G\tilde{y})$ and a vector in $T_{\tilde{y}}\tilde{U}$). This relation implies that the vector field $\widetilde{X}^x \in \mathcal{V}(\tilde{U})$ given by

$$\widetilde{X}^x_{\tilde{y}} := \mathrm{pr}^G_{T_{\tilde{y}}\tilde{U}} X_{\tilde{y}} \text{ for } \tilde{y} \in \tilde{U} \tag{2.3}$$

is Γ-invariant.

The various vector fields $\widetilde{X}^x$ for $x \in M/G$ induce an orbifold vector field on M/G: Let $x_1, x_2 \in M/G$ and choose $\tilde{x}_i \in P^{-1}(x_i)$, $i = 1,2$, together with charts $(U_i, \tilde{U}_i/\Gamma_i, \pi_i) = (B^{\varepsilon_i}/G, \psi_i^{-1}(\tilde{x}_i)/G_{\tilde{x}_i}, P_{|\psi^{-1}(\tilde{x}_i)})$ around x_i as above such that $U_1 \cap U_2 \neq \emptyset$ and let $\widetilde{X}^i := \widetilde{X}^{x_i} \in \mathcal{V}(\tilde{U}_i)^{\Gamma_i}$ be given by (2.3).

Now let $y \in U_1 \cap U_2$ be arbitrary and choose $\tilde{y}_i \in \pi_i^{-1}(y)$, h_i, $\hat{\psi}$, λ_i as in the compatibility proof of Thm. 2.2.1. We have to show that $\lambda_1^* \widetilde{X}^1 = \lambda_2^* \widetilde{X}^2$ on $\hat{\psi}^{-1}(\tilde{y}_1)$.

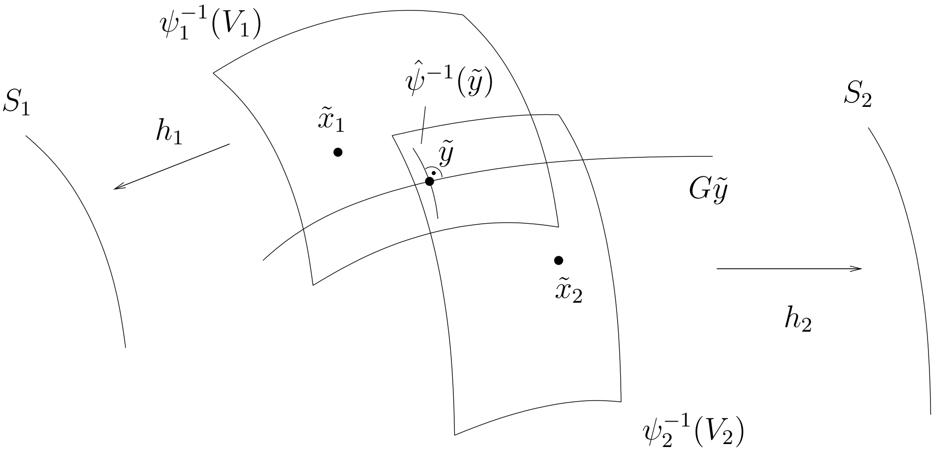
To this end set $\widehat{X}_{\tilde{r}} := \mathrm{pr}^G_{T_{\tilde{r}}\hat{\psi}^{-1}(\tilde{y}_1)} X_{\tilde{r}}$ for $\tilde{r} \in \hat{\psi}^{-1}(\tilde{y}_1)$. We claim that $\lambda_i^* \widetilde{X}^i = \widehat{X}$.

Recall that λ_i preserves G-orbits and that both $\widetilde{X}^i$ and $\widehat{X}$ (on their respective domains of definition) have the same G-horizontal components as X. Since X is G-invariant, $\lambda_i^* \widetilde{X}^i$ and $\widehat{X}$ have the same G-horizontal components. Since $\hat{\psi}^{-1}(\tilde{y}_1)$ is transverse to the vertical directions and both $\lambda_i^* \widetilde{X}^i$ and $\widehat{X}$ are tangent to $\hat{\psi}^{-1}(\tilde{y}_1)$, they must coincide.

Hence we obtain a well-defined vector field $\Phi(X) \in \mathcal{V}(M/G)$ induced by $\{\widetilde{X}^x\}_{x \in M/G}$. This map $\Phi\colon \overline{\mathcal{V}}^G_g(M) \to \mathcal{V}(M/G)$ is easily seen to be $C^\infty(M/G)$-linear.

The inverse of Φ: For the inverse let $Y \in \mathcal{V}(M/G)$. For $x \in M/G$ choose $\tilde{x} \in P^{-1}(x)$ and a chart of the form $(U, \widetilde{U}/\Gamma, \pi) = (B^\varepsilon(G\tilde{x})/G, \psi^{-1}(\tilde{x})/G_{\tilde{x}}, P_{|\psi^{-1}(\tilde{x})})$ around x and let $\widetilde{Y} := Y_\pi \in \mathcal{V}(\widetilde{U})^\Gamma$ be the corresponding vector field on $\widetilde{U}$ given by Y. Choose a neighbourhood V around $\tilde{x}$ in $G\tilde{x}$ and a submersion $h\colon \psi^{-1}(V) \to S$ as in (2) in the proof of Theorem 2.2.1. Recall that since the elements of G induce isometries between the horizontal spaces in M, we can define a Riemannian metric on S such that h becomes a Riemannian submersion. Then define X on $\psi^{-1}(V)$ as the horizontal lift of $(h_{|\widetilde{U}})_* \widetilde{Y} \in \mathcal{V}(S)$ via h. Note that $X_{\tilde{x}} = \widetilde{Y}(\tilde{x})$ but X does not in general coincide with $\widetilde{Y}$ on all of $\psi^{-1}(\tilde{x})$.

The compatibility conditions imply that the local vector fields for different $\tilde{x}$ patch together to give a smooth horizontal vector field X on M: Let $\tilde{x}_1, \tilde{x}_2 \in M$. For $i = 1, 2$ let ε_i, ψ_i, π_i, V_i, h_i corresponding to $\tilde{x}_i$ be chosen as above and let $X^i \in \mathcal{V}(\psi^{-1}(V_i))$ be the corresponding vector field induced by $\widetilde{Y}^i \in \mathcal{V}(\psi^{-1}(\tilde{x}_i))^{\Gamma_i}$. Moreover, assume $\psi_1^{-1}(V_1) \cap \psi_2^{-1}(V_2) \neq \emptyset$ and let $\tilde{y}$ be an element of this intersection. We have to show that $X^1_{\tilde{y}} = X^2_{\tilde{y}}$.



Around $P(\tilde{y}) \in M/G$ there is a chart of the form $(B^\delta(G\tilde{y})/G, \hat{\psi}^{-1}(\tilde{y})/G_{\tilde{y}}, P_{|\hat{\psi}^{-1}(\tilde{y})})$ with $\hat{\psi}^{-1}(\tilde{y}) \subset \psi_1^{-1}(V_1) \cap \psi_2^{-1}(V_2)$. Set $\hat{\pi} := P_{|\hat{\psi}^{-1}(\tilde{y})}$ and recall from the proof of Theorem

2.2.1 that for $i = 1, 2$ the map

$$\lambda_i := \left(h_{i|\psi_i^{-1}(\tilde{x}_i)}\right)^{-1} \circ h_{i|\hat{\psi}^{-1}(\tilde{y})}$$

is an injection from $\hat{\pi}$ to π_i. Hence we have $\lambda_1^* \widetilde{Y}^1 = \lambda_2^* \widetilde{Y}^2 \in \mathcal{V}(\hat{\psi}^{-1}(\tilde{y}))^{G_{\tilde{y}}}$. We denote this vector field by $\widehat{Y}$ and note that since $T_{\tilde{y}}\hat{\psi}^{-1}(\tilde{y})$ is orthogonal to $T_{\tilde{y}}(G\tilde{y})$ in $T_{\tilde{y}}M$, the vector $\widehat{Y}_{\tilde{y}}$ is G-horizontal. Moreover, by the definition of $\widehat{Y}$,

$$h_{i*}\widehat{Y}_{\tilde{y}} = \left(h_{i|\hat{\psi}^{-1}(\tilde{y})}\right)_* \widehat{Y}_{\tilde{y}} = \left(h_{i|\psi_i^{-1}(\tilde{x}_i)}\right)_* \lambda_{i*}\widehat{Y}_{\tilde{y}} = \left(h_{i|\psi_i^{-1}(\tilde{x}_i)}\right)_* \widetilde{Y}^i_{\lambda_i(\tilde{y})} = h_{i*}X^i_{\tilde{y}},$$

where the last equality follows from $h_i(\lambda_i(\tilde{y})) = h_i(\tilde{y})$ and the definition of X^i. Since $X^i_{\tilde{y}}$, $i = 1, 2$, are also horizontal, we deduce that $X^1_{\tilde{y}} = \widehat{Y}_{\tilde{y}} = X^2_{\tilde{y}}$. Hence the vector field X is well-defined on M.

To see that X is G-invariant let $\tilde{x} \in M$ and cover $G\tilde{x}$ with finitely many neighbourhoods V_i around points $g_i\tilde{x}$ in $G\tilde{x}$ as above. Then for each i set $W_i := \{g \in G;\ gg_i\tilde{x} \in V_i\}$. This is an open neighbourhood of the identity in G and hence so is the finite intersection $W := \bigcap_i W_i$. Note that by the definition of X we have

$$g_* X_{\tilde{y}} = X_{g\tilde{y}}\ \forall g \in W, \tilde{y} \in G\tilde{x}.$$

Since W generates the group G, the equation above also holds for all $g \in G, \tilde{y} \in G\tilde{x}$, in particular $g_* X_{\tilde{x}} = X_{g\tilde{x}}\ \forall g \in G$. Since $\tilde{x} \in M$ was chosen arbitrarily, we deduce that X is G-invariant and set $\Psi(Y) := X$. This gives a $C^\infty(M/G)$-linear map $\Psi\colon\ \mathcal{V}(M/G) \to \overline{\mathcal{V}}_g^G(M)$.

It is not hard to see that indeed Ψ is the inverse of Φ: First, to see that $\Phi \circ \Psi(Y) = Y$ for all $Y \in \mathcal{V}(M/G)$, set $X := \Psi(Y)$ as above. Let $x \in M/G$ and let $(U, \tilde{U}/\Gamma, \pi) = (B^\varepsilon(G\tilde{x})/G, \psi^{-1}(\tilde{x})/G_{\tilde{x}}, P_{|\psi^{-1}(\tilde{x})})$ be a chart around x as above.

By definition of Ψ $X_{|\tilde{U}}$ is jsut the G-horizontal component of Y_π. By definition of Φ, it follows that $[\Phi(X)]_\pi$ is again the vector field Y_π tangent to $\tilde{U}$. Since $x \in M/G$ was abitrary, we deduce that $\Phi(\Psi(Y)) = Y$.

It remains to show that Φ is injective. But if $\Phi(X) = 0$ then, by construction, X must have been not only horizontal but also vertical, hence zero.

Relation between Φ and P_*: To show the last statement of the theorem, let $X \in \overline{\mathcal{V}}_g^G(M)$. Let $\tilde{x} \in M_G$ and choose $\varepsilon > 0$ so small that the chart $\pi := P_{|\psi^{-1}(\tilde{x})}\colon\ \psi^{-1}(\tilde{x}) \to B^\varepsilon(G\tilde{x})/G$ on M/G (in the notation of 2.2.1) is a diffeomorphism.

Since $X_{\tilde{x}}$ is G-horizontal and thus tangent to $\psi^{-1}(\tilde{x})$ in $\tilde{x}$, we have $X_{\tilde{x}} = [\Phi(X)_\pi]_{\tilde{x}}$, hence

$$P_*(X_{\tilde{x}}) = \pi_* X_{\tilde{x}} = \pi_*[\Phi(X)_\pi]_{\tilde{x}} = \Phi(X)^{\mathrm{reg}}_{\pi(\tilde{x})} = \Phi(X)^{\mathrm{reg}}_{P(\tilde{x})},$$

as claimed. Hence the vector field $P_*(X_{|M_G})$ on the manifold $(M/G)^{\mathrm{reg}}$ corresponds to the orbifold tensor field on $(M/G)^{\mathrm{reg}}$ induced by X via the isomorphism $\Phi\colon\ \overline{\mathcal{V}}_g^G(M) \to \mathcal{V}(M/G)$ defined above. □

We now come to the situation of Theorem 2.2.4 to note under which conditions a covariant tensor field on $\mathcal{O}$ induces a tensor field on $\mathcal{O}/G$.

Proposition 2.3.16. *Let $\mathcal{O}$, G and $P\colon \mathcal{O} \to \mathcal{O}/G$ be as in Theorem 2.2.4 and let $\tau \in T^{0,k}(\mathcal{O})^G$ be a G-invariant tensor field on $\mathcal{O}$. Then there is a unique tensor field $\bar{\tau} \in T^{0,k}(\mathcal{O}/G)$ such that $P^*\bar{\tau} = \tau$*

Proof. Since in the proof of Theorem 2.2.4 around each $x \in \mathcal{O}$ a possible lift of P with respect to the $\mathcal{O}$-chart $(U, \tilde{U}/\Gamma, \pi)$ and the corresponding $\mathcal{O}/G$-chart $(U/G_x, \tilde{U}/\tilde{G}_x, \sigma\circ\pi)$ is given by the identity on $\tilde{U}$, we must define $\bar{\tau}_{\sigma\circ\pi} = \tau_\pi$. But this indeed defines a tensor field on $\mathcal{O}/G$, since τ is G-invariant and hence τ_π is $\tilde{G}_x$-invariant. Compatibility of the local tensor fields is clear, since we can use the same injections for $\mathcal{O}/G$ as for $\mathcal{O}$. $\square$

We now restrict our attention to Riemannian metrics and make the following definition.

Definition 2.3.17. A *Riemannian covering* between two Riemannian orbifolds $(\mathcal{O}_1, g_1)$, $(\mathcal{O}_2, g_2)$ is an orbifold covering map $P\colon \mathcal{O}_1 \to \mathcal{O}_2$ which satisfies $P^*g_2 = g_1$.

We can now rephrase the proposition above for metrics:

Corollary 2.3.18. *Let $\mathcal{O}$, G and $P\colon \mathcal{O} \to \mathcal{O}/G$ be as in Theorem 2.2.4 and let g be a G-invariant Riemannian metric on $\mathcal{O}$. Then there is a unique Riemannian metric $\bar{g}$ on $\mathcal{O}/G$ such that $P\colon (\mathcal{O}, g) \to (\mathcal{O}/G, \bar{g})$ becomes a Riemannian covering.*

2.3.3 Fundamental Vector Fields

In this section we will define fundamental vector fields associated to a smooth Lie group action on an orbifold (cf. 2.3.13).

So let G be a Lie group acting smoothly and effectively on the orbifold $\mathcal{O}$ via an action ϕ and let $X \in T_eG$ be an element of the corresponding Lie algebra. We define a vector field X^* on $\mathcal{O}$ in the following way: Let $x \in \mathcal{O}$ be arbitrary. Since the action is smooth (cf. Definition 2.1.15), there are charts $(U, \tilde{U}/\Gamma, \pi)$ and $(U', \tilde{U}'/\Gamma', \pi')$ of $\mathcal{O}$ over x, an open neighbourhood W of e in G and a smooth map $\tilde{\phi}\colon W \times \tilde{U} \to \tilde{U}'$ such that the following diagram commutes.

$$\begin{tikzcd} W \times \tilde{U} \arrow[r, "\tilde{\phi}"] \arrow[d, "(\mathrm{id},\pi)"] & \tilde{U}' \arrow[d, "\pi'"] \\ W \times U \arrow[r, "\phi"] & U' \end{tikzcd}$$

Moreover, we can assume that there is a homomorphism $\Theta\colon \Gamma \to \Gamma'$ such that

$$\tilde{\phi}(g, \gamma\tilde{y}) = \Theta(\gamma)\tilde{\phi}(g, \tilde{y}) \;\forall g \in W, \gamma \in \Gamma, \tilde{y} \in \tilde{U}. \tag{2.4}$$

For $g \in W$, $\tilde{y} \in \tilde{U}$ set $\tilde{\phi}^g(\tilde{y}) := \tilde{\phi}(g, \tilde{y})$. Then $h := \tilde{\phi}^e$ is a covering onto its image by [MM03] Prop. 2.13 because $\pi' \circ h = \pi$. By choosing U sufficiently small around x, we can assume that $h\colon \tilde{U} \to \tilde{U}'$ is an embedding. Denote the inverse $\tilde{U}' \supset h(\tilde{U}) \to \tilde{U}$ by h^{-1}.

Moreover, for $\tilde{y} \in \tilde{U}$ define $\tilde{\phi}^{\tilde{y}}\colon W \to \tilde{U}'$ by $\tilde{\phi}^{\tilde{y}}(g) := \tilde{\phi}(g, \tilde{y})$. Recall that we had fixed $X \in T_eG$ and define a vector field $\sigma_x(X)$ on $\tilde{U}$ by

$$\begin{aligned}\sigma_x(X)(\tilde{y}) &:= dh^{-1}_{\tilde{y}} d\tilde{\phi}^{\tilde{y}}{}_{|e} X = dh^{-1}_{\tilde{y}} \frac{d}{dt}_{|t=0} \tilde{\phi}(\exp(tX), \tilde{y}) \\ &= \frac{d}{dt}_{|t=0} h^{-1}(\tilde{\phi}(\exp(tX), \tilde{y})),\end{aligned}$$

where exp denotes the Lie group exponential map.

$\sigma_x(X)$ is Γ-invariant because by applying (2.4) (and its implication $h \circ \gamma = \Theta(\gamma) \circ h$) we obtain

$$\begin{aligned}\sigma_x(X)(\gamma\tilde{y}) = \frac{d}{dt}_{|t=0} h^{-1}(\tilde{\phi}(\exp(tX), \gamma\tilde{y})) = \frac{d}{dt}_{|t=0} h^{-1}(\Theta(\gamma) \circ \tilde{\phi}(\exp(tX), \tilde{y})) \\ = \frac{d}{dt}_{|t=0} \gamma \circ h^{-1}(\tilde{\phi}(\exp(tX), \tilde{y})) = \gamma_* \sigma_x(X)(\tilde{y}).\end{aligned}$$

Moreover, note that $\sigma_x(X)$ does not depend on the choice of the lift $\tilde{\phi}$ of the G-action ϕ: Given two lifts $\tilde{\phi}_i\colon W \times \tilde{U} \to \tilde{U}'_i$, $i = 1, 2$, of $\phi_i\colon W \times U \to U'_i$ (where each ϕ_i is just a restriction of ϕ) and a regular point $\tilde{y}$ in $\tilde{U}$, the paths $c_i\colon (-\varepsilon, \varepsilon) \ni t \mapsto h_i^{-1}(f_i(\exp(tX), \tilde{y})) \in \tilde{U}$ (with $\varepsilon > 0$ so small that $\exp(tX) \in W$ for all $t \in (-\varepsilon, \varepsilon)$) satisfy $c_i(0) = \tilde{y}$ and $\pi \circ c_1 = \pi \circ c_2$. Since $\tilde{y}$ is regular, the two paths coincide on an open neighbourhood of 0, hence we have $\dot{c}_1(0) = \dot{c}_2(0)$ and $\sigma_x(X)(\tilde{y}) \in T_{\tilde{y}}\tilde{U}$ is independent of f. Since the set of regular points is dense in $\tilde{U}$, $\sigma_x(X)$ does not depend on the lift f but only on π and the given G-action on $\mathcal{O}$.

We can define such a local vector field $\sigma_x(X) \in \mathcal{V}(\tilde{U})^\Gamma$ for every $x \in \mathcal{O}$ as above. It remains to show that $\{\sigma_x(X)\}_{x \in \mathcal{O}}$ induces a smooth orbifold vector field X^* on $\mathcal{O}$.

Since, in the definition of $\sigma_x(X)$, the neighbourhood U of x can be chosen arbitrarily small, it suffices to show that for charts $(U_i, \tilde{U}_i/\Gamma_i, \pi_i)$, $i = 1, 2$, with $U_1 \subset U_2$ and $\sigma_i(X) \in \mathcal{V}(\tilde{U}_i)^{\Gamma_i}$ defined as above, we have for an injection λ from π_1 to π_2:

$$\lambda_*(\sigma_1(X)(\tilde{y})) = \sigma_2(X)(\lambda(\tilde{y}))\ \forall \tilde{y} \in \tilde{U}_1. \tag{2.5}$$

To see this note that if $\tilde{\phi}_2\colon W \times \tilde{U}_2 \to \tilde{U}'$ is a lift of the G-action ϕ on U_2 with respect to π_2 and a chart $(U', \tilde{U}'/\Gamma', \pi')$ (which exists since ϕ is smooth, see the diagram above), then $\tilde{\phi}_1 := \tilde{\phi}_2 \circ (\mathrm{id}, \lambda)$ is a lift of ϕ on U_1 with respect to π_1 and π'. Moreover, note that we have assumed that $h_i := \tilde{\phi}_i^e$ are embeddings. Using $h_1 = h_2 \circ \lambda$ we calculate

$$\begin{aligned}\sigma_2(X)(\lambda(\tilde{y})) &= \frac{d}{dt}_{|t=0} h_2^{-1}(\tilde{\phi}_2(\exp(tX), \lambda(\tilde{y}))) = \frac{d}{dt}_{|t=0} \lambda \circ h_1^{-1}(\tilde{\phi}_1(\exp(tX), \tilde{y})) \\ &= \lambda_* \sigma_1(X)(\tilde{y}).\end{aligned}$$

Hence X^* is indeed an orbifold vector field.

Definition 2.3.19. Let G be a Lie group acting effectively on an orbifold $\mathcal{O}$.

(i) For $X \in T_e G$ the vector field $X^* \in \mathcal{V}(\mathcal{O})$ defined above is called the *fundamental vector field* on $\mathcal{O}$ associated with X.

(ii) A 1-form $\lambda \in \Omega^1(\mathcal{O})$ is called *G-horizontal* if $\lambda(X^*) = 0 \in C^\infty(\mathcal{O})$.

Note that this definition is coherent with the usual definition in the case that $\mathcal{O}$ is a manifold, where λ is called G-horizontal if it vanishes on vectors tangent to the G-orbits.

Moreover, recall that for an abelian Lie group acting on a manifold M we have that the (ordinary) fundamental vector fields $X^*\colon M \ni p \mapsto \frac{d}{dt}_{|t=0} \exp(tX)p \in TM$ are G-invariant, i.e., $g_* X^* = X^*$ or, more precisely,

$$dg_p X_p^* = X_{gp}^* \ \forall p \in M, X \in T_e G, g \in G.$$

The following lemma provides a generalization to orbifolds in a form which we will need in a later chapter. Recall from Lemma 2.1.19 that orbifold diffeomorphisms locally lift to manifold diffeomorphisms, i.e., charts and lifts as in the following lemma indeed exist around every point of $\mathcal{O}$.

Lemma 2.3.20. *Let G be an abelian Lie group acting on an orbifold $\mathcal{O}$. If $(U_i, \tilde{U}_i/\Gamma_i, \pi_i)$, $i = 1, 2$, are charts on $\mathcal{O}$, $g \in G$ and $\tilde{g}\colon \tilde{U}_1 \to \tilde{U}_2$ is a diffeomorphism such that $\pi_2 \circ \tilde{g} = g \circ \pi_1$, then for all $X \in T_e G$:*

$$\tilde{g}_* X_{\pi_1}^* = X_{\pi_2}^*.$$

Proof. Let $\tilde{x} \in \tilde{U}_1$ and set $x := \pi_1(\tilde{x})$. Since it suffices to show the relation above in $\tilde{x}$, we can assume that U_1 and $\tilde{U}_1$ are so small that there is a lift $\tilde{\phi}_1$ of the restriction $\phi_1\colon W \times U_1 \to U'$ of the G-action ϕ, i.e., there is a commutative diagram of the following form (and a homomorphism Θ_1 associated to $\tilde{\phi}_1$ as in Def. 2.1.15):

$$\begin{tikzcd} W \times \tilde{U}_1 \arrow[r, "\tilde{\phi}_1"] \arrow[d, "(\mathrm{id}, \pi_1)"'] & \tilde{U}' \arrow[d, "\pi'"] \\ W \times U_1 \arrow[r, "\phi_1"] & U' \end{tikzcd}$$

Moreover, we can assume that $\tilde{\phi}_1^e =: h_1$ is an embedding and without loss of generality that $g(U_1) = U_2$, $\tilde{g}(\tilde{U}_1) = \tilde{U}_2$, $\Gamma_2 = \tilde{g}\Gamma_1\tilde{g}^{-1}$.

Now set $\tilde{\phi}_2 := \tilde{\phi}_1 \circ (\mathrm{id}_W, \tilde{g}^{-1})$. Then, since G is abelian, we have $\phi(W \times U_2) \subset gU'$ and the following diagram commutes:

$$\begin{tikzcd} W \times \tilde{U}_2 \arrow[r, "\tilde{\phi}_2"] \arrow[d, "(\mathrm{id},\pi_2)"'] & \tilde{U}' \arrow[d, "g\circ\pi'"] \\ W \times U_2 \arrow[r, "\phi_2"] & gU' \end{tikzcd}$$

Now, recall that Θ_1 denotes a homomorphism associated to the lift $\tilde{\phi}_1$. Then $\Theta_2(\gamma) := \Theta_1(\tilde{g}^{-1} \circ \gamma \circ \tilde{g})$ gives a homomorphism Θ_2 associated to $\tilde{\phi}_2$, i.e., $\tilde{\phi}_2$ is indeed a lift of the restriction ϕ_2 of the G-action in the sense of Definition 2.1.15. Morever, for $\tilde{\phi}_2^e =: h_2$ we have $h_2 = h_1 \circ \tilde{g}^{-1}$, i.e., h_2 is again an embedding and we can calculate:

$$\begin{aligned} \tilde{g}_* X^*_{\pi_1}(\tilde{x}) &= \tilde{g}_* \frac{d}{dt}_{|t=0} h_1^{-1}(\tilde{\phi}_1(\exp(tX), \tilde{x})) \\ &= \frac{d}{dt}_{|t=0} (\tilde{g} \circ h_1^{-1}) \circ \tilde{\phi}_2(\exp(tX), \tilde{g}(\tilde{x})) \\ &= \frac{d}{dt}_{|t=0} h_2^{-1}(\tilde{\phi}_2(\exp(tX), \tilde{g}(\tilde{x}))) \\ &= X^*_{\pi_2}(\tilde{g}(\tilde{x})) \end{aligned}$$

□

2.4 Integration

2.4.1 Integration on Oriented Orbifolds

Let $\mathcal{O}$ be a compact oriented n-dimensional orbifold with an oriented (not necessarily maximal) atlas $\{(U_\alpha, \tilde{U}_\alpha, /\Gamma_\alpha, \pi_\alpha)\}$ and let $\omega = \{\omega_\alpha\}$ be an n-form on $\mathcal{O}$. Then choose a finite subcovering $\{U_{\alpha(i)}\} \subset \{U_\alpha\}$ of $\mathcal{O}$ and a smooth partition of unity $\{\psi_i\}$ such that each ψ_i has support in $U_{\alpha(i)}$ (cf. [Chi90]). The integral of the n-form ω is then defined by

$$\int_{\mathcal{O}} \omega := \sum_i \frac{1}{|\Gamma_{\alpha(i)}|} \int_{\tilde{U}_{\alpha(i)}} (\psi_i \circ \pi_{\alpha(i)}) \omega_{\alpha(i)}. \tag{2.6}$$

This definition can be shown to be independent of the chosen covering and the partition of unity (cf. [Wei07] 2.2).

Lemma 2.4.1. *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two compact oriented n-dimensional orbifolds, $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ is an orientation-preserving diffeomorphism and ω is an n-form on $\mathcal{O}_2$, then*

$$\int_{\mathcal{O}_1} f^*\omega = \int_{\mathcal{O}_2} \omega.$$

Proof. Let $\{(U_\alpha, \tilde{U}_\alpha/\Gamma_\alpha, \pi_\alpha)\}_\alpha$ be a positively oriented atlas on $\mathcal{O}_2$ and note that then $\{(f^{-1}(U_\alpha), \tilde{U}_\alpha/\Gamma_\alpha, f^{-1} \circ \pi_\alpha)\}_\alpha$ is a positively oriented atlas on $\mathcal{O}_1$. Choose a finite subcovering $\{U_{\alpha(i)}\}$ and an associated partition of unity $\{\psi_i\}$ on $\mathcal{O}_2$ and write index i

instead of $\alpha(i)$. Then we have (since $\{\psi_i \circ f\}_i$ is a partition of unity on $\mathcal{O}_1$ and $f^*\omega$ in the $\mathcal{O}_1$-chart $f^{-1} \circ \pi_i$ is given by $\omega_i := \omega_{\pi_i}$):

$$\begin{aligned}\int_{\mathcal{O}_1} f^*\omega &= \sum_i \frac{1}{|\Gamma_i|} \int_{\tilde{U}_i} ((\psi_i \circ f) \circ (f^{-1} \circ \pi_i))(f^*\omega)_{f^{-1}\circ\pi_i} \\ &= \sum_i \frac{1}{|\Gamma_i|} \int_{\tilde{U}_i} (\psi_i \circ \pi_i)\omega_i = \int_{\mathcal{O}_2} \omega\end{aligned}$$

□

Now let in addition g be a Riemannian metric on $\mathcal{O}$. The *volume form* dvol_g on $\mathcal{O}$ is the n-form given by the local volume forms $\{\mathrm{dvol}_{g_\alpha}\}$ in an oriented atlas of $\mathcal{O}$. The integral of a function $f \in C^\infty(\mathcal{O})$ is then given by

$$\int_{\mathcal{O}} f := \int_{\mathcal{O}} f \,\mathrm{dvol}_g\,.$$

As usual, we set $\mathrm{vol}(\mathcal{O}) = \int_{\mathcal{O}} 1$, and the Hilbert space $L^2(\mathcal{O})$ is the completion of $C^\infty(\mathcal{O})$ with respect to the scalar product

$$(f_1, f_2) = (f_1, f_2)_0 = \int_{\mathcal{O}} f_1 f_2.$$

2.4.2 Integration on Non-Oriented Orbifolds

To phrase our theorems in Chapter 4 for not necessarily oriented orbifolds, pwe will need to define integrals of functions on compact orbifolds independently from orientation. We will use the following approach (cf. [Lee03], Chapter 14).

Definition 2.4.2. (i) Let V be an n-dimensional real vector space. A *density* on V is a function $\mu\colon V^n \to \mathbb{R}$ such that for all endomorphisms T on V we have

$$\mu(TX_1, \ldots, TX_n) = |\det T| \mu(X_1, \ldots, X_n)\ \forall X_1, \ldots X_n \in V.$$

(ii) If M is a smooth n-dimensional manifold, the *density bundle* ΩM is the disjoint union of the vector spaces of densities on all $T_pM,\ p \in M$.

ΩM in the definition above can be given the structure of a smooth line bundle over M and sections of ΩM are called densities on M. If μ is a section on a manifold M, the integral $\int_M \mu$ of a density μ with compact support can be defined by first setting $\int_U f|dx^1 \wedge \ldots \wedge dx^n| := \int_V f dx^1 \ldots dx^n$ for $x\colon U \to V$ a chart on M and $f \in C_0^\infty(U)$ and then defining the integral of μ on M via a partition of unity.

Note that, using the same formula as in the case of forms, densities can be pulled back by smooth maps between manifolds. This enables us to define densities on orbifolds:

Definition 2.4.3. Let $\mathcal{O}$ be an orbifold. A *density* μ on $\mathcal{O}$ is an assignment of a Γ-invariant density $\mu_\pi \in \Omega(\tilde{U})^\Gamma$ to every $\mathcal{O}$-chart $(U, \tilde{U}/\Gamma, \pi)$ such that:

For $(U_i, \tilde{U}_i/\Gamma_i, \pi_i), i = 1, 2$, charts on $\mathcal{O}$ and $x \in U_1 \cap U_2$ there is an $\mathcal{O}$-chart $(U, \tilde{U}/\Gamma, \pi)$ and injections $\lambda_i, i = 1, 2$, from π to π_i satisfying $x \in U \subset U_1 \cap U_2$ such that $\lambda_1^* \mu_{\pi_2} = \lambda_2^* \mu_{\pi_2} \in \Omega(\tilde{U})^\Gamma$.

A proof analogous to that of Lemma 2.3.2 shows that a covering of an orbifold $\mathcal{O}$ by charts together with corresponding densities compatible in the sense of the definition above induce a density on $\mathcal{O}$. Using this we can define the pull-back of densities via smooth orbifold maps analogously to Lemma 2.3.7.

The integral of a density μ on a compact and not necessarily oriented orbifold $\mathcal{O}$ is then defined by the formula (2.6) with ω replaced by μ (and the atlas not assumed to be oriented, of course). The proof of Lemma 2.4.1 then carries over to the case of densities, i.e., we have:

Lemma 2.4.4. *Given two compact orbifolds $\mathcal{O}_1$, $\mathcal{O}_2$, a diffeomorphism $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ and a density μ on $\mathcal{O}_2$, we have*

$$\int_{\mathcal{O}_1} f^* \mu = \int_{\mathcal{O}_2} \mu.$$

To integrate functions we use the fact that every Riemannian manifold (M, g) carries a unique density μ such that $\mu(E_1, \ldots, E_n) = 1$ for every local orthonormal frame $\{E_1, \ldots, E_n\}$ on (M, g). We will denote this density by dvol_g. The Riemannian density dvol_g on a Riemannian orbifold $(\mathcal{O}, g)$ is then given by the density $\mathrm{dvol}_g := \{\mathrm{dvol}_{g_\alpha}\}_{\alpha \in I(\mathfrak{A})}$ with $\mathfrak{A}$ a maximal atlas on $\mathcal{O}$. The integral of a function $f \in C_0^\infty(\mathcal{O})$ over $(\mathcal{O}, g)$ is defined by $\int_{\mathcal{O}} f := \int_{\mathcal{O}} f \,\mathrm{dvol}_g$. Since in the case of an oriented Riemannian manifold (M, g) the two definitions of integrals of functions over (M, g) coincide, so do the two definition of integrals of functions on compact oriented Riemannian orbifolds given in this section. In this work dvol will always denote the Riemannian density unless otherwise stated.

3 Spectral Geometry on Orbifolds

3.1 The Spectrum

Definition 3.1.1. Let $(\mathcal{O}, \langle , \rangle)$ be a compact Riemannian orbifold.

(i) The *Laplace operator* $\Delta\colon C^\infty(\mathcal{O}) \to C^\infty(\mathcal{O})$ is given in the following way: For $f \in C^\infty(\mathcal{O})$ and $x \in \mathcal{O}$ let $(U, \tilde{U}/\Gamma, \pi)$ be a chart around x and set $\Delta f(x) := \tilde{\Delta}(f \circ \pi)(\tilde{x})$, where $\tilde{x} \in \pi^{-1}(x)$ and $\tilde{\Delta} = \delta d$ denotes the (non-negative) Laplacian on the manifold $\tilde{U}$ with the Riemannian metric given by $\langle , \rangle$.

(ii) The *spectrum* $\operatorname{spec}(\mathcal{O})$ is the set of eigenvalues of Δ with (finite, see below) multiplicities, i.e., $\operatorname{spec}(\mathcal{O}) \subset \mathbb{R}$ is a multiset, where the multiplicity of $\mu \in \operatorname{spec}(\mathcal{O})$ is the dimension of the eigenspace

$$E_\mu(\mathcal{O}) := \{f \in C^\infty(\mathcal{O});\ \Delta f = \mu f\}$$

of Δ to the eigenvalue μ.

(iii) Two compact Riemannian orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ are called *isospectral* if $\operatorname{spec}(\mathcal{O}_1) = \operatorname{spec}(\mathcal{O}_2)$ with multiplicities.

The spectrum of the Laplacian on compact orbifolds was first investigated by Donnelly ([Don79]). He proved the following theorem for good orbifolds which was later generalized to arbitrary orbifolds by Chiang ([Chi90] Prop. 3.2), also compare [DGGW08].

Theorem 3.1.2. *Let $(\mathcal{O}, \langle , \rangle)$ be a compact Riemannian orbifold. Then every eigenvalue of Δ on $C^\infty(\mathcal{O})$ has finite multiplicity and $\operatorname{spec}(\mathcal{O})$ consists of a sequence $0 = \mu_0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \ldots$, where $\mu_i \to \infty$. Moreover there is an orthonormal basis $\{\phi_i\}_{i \geqslant 0} \subset C^\infty(\mathcal{O})$ of the Hilbert space $L^2(\mathcal{O})$ such that $\Delta\phi_i = \mu_i\phi_i$.*

From now on, we assume that all orbifolds are compact and Riemannian. Recall that for $X, Y \in \mathcal{V}(\mathcal{O})$ we had defined $\langle X, Y\rangle \in C^\infty(\mathcal{O})$ by $\langle X, Y\rangle(x) = \langle X_\pi, Y_\pi\rangle_\pi(\tilde{x})$ with π a chart around $x \in \mathcal{O}$, and $\tilde{x} \in \pi^{-1}(x)$. We will need Green's Formula for orbifolds. Note that we do not assume $\mathcal{O}$ to be oriented.

Lemma 3.1.3. *Let $\mathcal{O}$ be a compact Riemannian orbifold and let $f_i \in C^\infty(\mathcal{O}), i = 1, 2$. Then*

$$\int_{\mathcal{O}} f_1 \Delta f_2 = \int_{\mathcal{O}} \langle \operatorname{grad} f_1, \operatorname{grad} f_2\rangle.$$

Proof. Let $\{U_i\}$ be a finite covering of $\mathcal{O}$ with associated charts $\{U_i, \tilde{U}_i/\Gamma_i, \pi_i)\}_i$ and let $\{\psi_i\}_i$ be a subordinate partition of unity. First, by the definition of the Laplacian on $\mathcal{O}$:

$$\int_{\mathcal{O}} f_1 \Delta f_2 = \sum_i \int_{\mathcal{O}} \psi_i f_1 \Delta f_2 = \sum_i \frac{1}{|\Gamma_i|} \int_{\tilde{U}_i} (\psi_i f_1 \Delta f_2) \circ \pi_i$$
$$= \sum_i \frac{1}{|\Gamma_i|} \int_{\tilde{U}_i} (\psi_i f_1) \circ \pi_i \tilde{\Delta}_i (f_2 \circ \pi_i)$$

Since $(\psi_i f_1) \circ \pi_i$ has compact support in $\tilde{U}_i$ (for each i), Green's Identity on the Riemannian manifold $\tilde{U}_i$ (which follows directly from the divergence theorem, cf. [Lee03] Thm. 14.34) implies (with $\langle , \rangle_i := \langle , \rangle_{\pi_i}$):

$$\int_{\mathcal{O}} f_1 \Delta f_2 = \sum_i \frac{1}{|\Gamma_i|} \int_{\tilde{U}_i} \langle \operatorname{grad}((\psi_i f_1) \circ \pi_i), \operatorname{grad}(f_2 \circ \pi_i) \rangle_i. \tag{3.1}$$

Moreover, note that, since ψ_i has compact support in U_i, we have:

$$\sum_i \frac{1}{|\Gamma_i|} \int_{\tilde{U}_i} f_1 \circ \pi_i \langle \operatorname{grad}(\psi_i \circ \pi_i), \operatorname{grad}(f_2 \circ \pi_i) \rangle_i = \sum_i \int_{\mathcal{O}} f_1 \langle \operatorname{grad} \psi_i, \operatorname{grad} f_2 \rangle$$
$$= \int_{\mathcal{O}} f_1 \langle \operatorname{grad}(\sum_i \psi_i), \operatorname{grad} f_2 \rangle = 0 \tag{3.2}$$

Now (3.1) and (3.2) imply

$$\int_{\mathcal{O}} f_1 \Delta f_2 = \sum_i \frac{1}{|\Gamma_i|} \int_{\tilde{U}_i} \langle \operatorname{grad}((\psi_i f_1) \circ \pi_i), \operatorname{grad}(f_2 \circ \pi_i) \rangle_i$$
$$= \sum_i \frac{1}{|\Gamma_i|} \int_{\tilde{U}_i} (\psi_i \circ \pi_i) \langle \operatorname{grad}(f_1 \circ \pi_i), \operatorname{grad}(f_2 \circ \pi_i) \rangle_i$$
$$= \int_{\mathcal{O}} \langle \operatorname{grad} f_1, \operatorname{grad} f_2 \rangle$$

□

Now consider $C^\infty(\mathcal{O})$ as a Pre-Hilbert-space with the inner product

$$(f, g)_1 = \int_{\mathcal{O}} fg + \int_{\mathcal{O}} \langle \operatorname{grad} f, \operatorname{grad} g \rangle.$$

The Sobolev space $H^1(\mathcal{O}) \subset L^2(\mathcal{O})$ is the completion of $C^\infty(\mathcal{O})$ with respect to this inner product. The Rayleigh quotient $R\colon H^1(\mathcal{O}) \backslash \{0\} \to [0, \infty)$ is the unique continuous extension of the functional

$$R\colon C^\infty(\mathcal{O}) \setminus \{0\} \ni f \mapsto \frac{\int \langle \operatorname{grad} f, \operatorname{grad} f \rangle}{\int f^2} \in [0, \infty)$$

to $H^1(\mathcal{O}) \setminus \{0\}$. Theorem 3.1.2 implies the following variational characterization.

Theorem 3.1.4. *Let $\mathcal{O}$ be a compact Riemannian orbifold and let L_k denote the set of all k-dimensional subspaces of $H^1(\mathcal{O})$. Then*

$$\mu_k = \inf_{U \in L_k} \sup_{f \in U \setminus \{0\}} R(f).$$

Proof. Using Theorem 3.1.2, the proof is literally the same as in the manifold case ([Bér86] III.28). We include it for completeness. Let $\{\phi_i\}$ and $\{\mu_i\}$ be chosen as in Theorem 3.1.2. Then we have for $f \in L^2(\mathcal{O})$:

$$f = \sum_{i=0}^{\infty} (f, \phi_i)_0 \phi_i \text{ and } \|f\|_0^2 = \sum_{i=0}^{\infty} (f, \phi_i)_0^2.$$

Now let $f \in H^1(\mathcal{O}) \setminus \{0\}$ and set $a_i = (f, \phi_i)_0$. We first show that

$$R(f) = \frac{\sum_{i \geqslant 0} \mu_i a_i^2}{\sum_{i \geqslant 0} a_i^2} \tag{3.3}$$

We can assume that $f \in C^\infty(\mathcal{O})$. Applying Lemma 3.1.3, we obtain:

$$\begin{aligned} R(f) &= \frac{\int \langle \operatorname{grad} f, \operatorname{grad} f \rangle}{\|f\|_0^2} = \frac{\int f \Delta f}{\sum_i a_i^2} = \frac{\int \sum_i (f, \phi_i)_0 \phi_i \sum_j (f, \phi_j)_0 \Delta \phi_j}{\sum_i a_i^2} \\ &= \frac{\sum_{i,j} (f, \phi_i)_0 (f, \phi_j)_0 \mu_i (\phi_i, \phi_j)_0}{\sum_i a_i^2} = \frac{\sum_i \mu_i a_i^2}{\sum_i a_i^2} \end{aligned}$$

To finish the proof of Theorem 3.1.4 set $\nu_k := \inf_{U \in L_k} \sup_{f \in U \setminus \{0\}} R(f)$.

- $\nu_k \leqslant \mu_k$: Choose $U = \operatorname{span}\{\phi_1, \ldots, \phi_k\}$. For $f \in U \setminus \{0\}$ equation (3.3) implies

  $$R(f) = \frac{\sum_{i=0}^k \mu_i a_i^2}{\sum_{i=0}^k a_i^2} \leqslant \frac{\sum_{i=0}^k \mu_k a_i^2}{\sum_{i=0}^k a_i^2} = \mu_k,$$

  hence $\nu_k \leqslant \mu_k$.

- $\nu_k \geqslant \mu_k$: Let $U \in L_k$ be arbitrary. There is $f \in U$ with $f \perp \phi_i \ \forall i \in \{1, \ldots, k-1\}$. Then (3.3) implies

  $$R(f) = \frac{\sum_{j=k}^\infty \mu_j a_j^2}{\sum_{j=k}^\infty a_j^2} \geqslant \frac{\sum_{j=k}^\infty \mu_k a_j^2}{\sum_{j=k}^\infty a_j^2} = \mu_k,$$

  hence $\sup_{f \in U \setminus \{0\}} R(f) \geqslant \mu_k$. Since $U \in L_k$ was arbitrary, we conclude that $\nu_k \geqslant \mu_k$.

□

Remark. Note that there are other similar characterizations of the eigenvalues on an orbifold like e.g. Rayleigh's Theorem in [Sta05] Lemma 6.3.

3.2 Isospectral Orbifolds

As in the manifold case it can be shown that the spectrum determines the volume, dimension and other geometric properties of an orbifold ([Far01], [DS09]). In order to investigate which properties are not determined by the spectrum, one needs constructions of isospectral (but non-isometric) orbifolds. There are some general constructions of isospectral manifolds (cf. [Gor00]), but in this section we concentrate on those which have already been generalized to get examples of isospectral singular orbifolds.

The following theorem goes back to Sunada ([Sun85]) and Berard ([Bér92]). The simplest proof for the manifold case (cf. [Gor00]) easily carries over to orbifolds.

Theorem 3.2.1. *Let (M, g) be a compact Riemannian manifold and let Γ_1, Γ_2 be two finite subgroups of the isometry group $\mathrm{Isom}(M)$ of M. Moreover, assume that there is a bijection $\phi\colon \Gamma_1 \to \Gamma_2$ such that $\phi(\gamma)$ is conjugate to γ in $\mathrm{Isom}(M)$ for every $\gamma \in \Gamma_1$. Then the orbifolds $(M/\Gamma_1, g)$, $(M/\Gamma_2, g)$ with the induced metrics are isospectral.*

This theorem was used to show that the spectrum does not determine the isomorphism classes of isotropy groups on an orbifold ([SSW06]). Both the Sunada Theorem and an explicit formula for eigenvalues on flat orbifolds ([MR03]) can even be used to construct pairs of isospectral orbifolds in which the maximal orders of isotropy groups are different ([RSW08]).

The following generalization of the Sunada Theorem was used in [PS08] to give continuous families of isospectral singular orbifolds.

Theorem 3.2.2. *Let G be a Lie group with simply connected and nilpotent identity component G_0. Let Γ be a discrete subgroup of G such that $(G_0 \cap \Gamma)\backslash G_0$ is compact and $G = \Gamma G_0$. Let Φ be an automorphism on G such that $\Phi(\gamma)$ is conjugate to γ in G for every $\gamma \in \Gamma$. Moreover, let (M, g) be a Riemannian manifold on which G acts effectively, properly and isometrically on the left with compact quotient $\Gamma\backslash M$. Then the orbifolds $(\Gamma\backslash M, g)$ and $(\Phi(\Gamma)\backslash M, g)$ are isospectral.*

There is another generalization of Sunada's Theorem which can be used to give isospectral orbifolds. First recall the following proposition (cf. [Kaw91] Thm. 4.27).

Proposition 3.2.3. *If G is a compact Lie group acting effectively on a connected manifold M, there is a unique conjugation class $[H]$ of subgroups of G (the so-called principal isotropy class or generic stabilizer) such that:*

(i) For every $x \in M$ the class $[H]$ contains a subgroup of the stabilizer G_x.

(ii) The set $\{x \in M;\ G_x \in [H]\}$ is open and dense in M.

In the situation of the proposition above we denote by τ^G the representation of G on $L^2(M)$ given by $\tau^G(g)f(x) = f(g^{-1}x)$

Now assume we are given an arbitrary representation of G on a vector space V. For a subgroup K of G set $V^K := \{v \in V;\ kv = v\ \forall k \in K\}$ and let V_K be the intersection of all G-invariant subspaces of V containing V^K. Note that conjugate subgroups K, gKg^{-1} of

G yield equivalent G-representations on V_K and $V_{gKg^{-1}}$, respectively, because $V_{gKg^{-1}} = gV_K$.

For a subgroup Γ of G consider the quasi-regular representation π_Γ^G on $L^2(G/\Gamma)$ given by $\pi_\Gamma^G(g)f(x) = f(g^{-1}x)$. Then, with $[K]$ a conjugation class of subgroups of G, two subgroups Γ_1, Γ_2 of G are called $[K]$-equivalent if the restrictions of $\pi_{\Gamma_1}^G$ and $\pi_{\Gamma_2}^G$ to $L^2(G/\Gamma_1)_K$ and $L^2(G/\Gamma_2)_K$, respectively, are equivalent G-representations.

The following theorem is then due to Sutton ([Sut06] Thm. 3.15) and generalizes ideas of Pesce ([Pes96]).

Theorem 3.2.4. *Let G be a compact Lie group acting effectively and isometrically on two compact Riemannian manifolds M_1, M_2 such that*

1. *M_1, M_2 are equivariantly isospectral with respect to G, i.e., there is a unitary isomorphism $U\colon L^2(M_1) \to L^2(M_2)$ between the G-representations τ_1^G and τ_2^G such that U maps eigenfunctions on M_1 to eigenfunctions on M_2 associated with the same eigenvalue.*

2. *The G-actions on M_1, M_2 have the same generic stabilizer $[K]$.*

Then if Γ_1, Γ_2 are $[K]$-equivalent subgroups of G, the orbifolds M_1/Γ_1 and M_2/Γ_2 are isospectral.

Applying this theorem to the constructions of [Gor01] and [Sch01], Sutton obtained for an arbitrary finite subgroup Γ of the 2-torus T (with the T-action given in the corresponding publication) isospectral families of metrics on the singular orbifold S^n/Γ ($n \geqslant 7$) and isospectral pairs on S^5/Γ. Besides, Sutton also gives a version for orbifolds with boundary. For a simple version of the theorem above for equivariantly isospectral orbifolds see Theorem 5.4.1

However, all the constructions above give pairs of good orbifolds, and to our best knowledge there are no publications on isospectral bad orbifolds. The only obvious way to construct isospectral bad (i.e. non-good) orbifolds using known constructions would be to take a pair of good isospectral orbifolds $\mathcal{O}_1, \mathcal{O}_2$ (which can of course be manifolds) and a bad orbifold $\mathcal{O}$. Then the Riemannian products $\mathcal{O}_1 \times \mathcal{O}$ and $\mathcal{O}_2 \times \mathcal{O}$ are isospectral bad orbifolds. However, in Chapter 5 we will present the first examples of isospectral bad orbifolds which cannot be written as non-trivial products.

3.3 Obstructions to Isospectrality

The central question of the spectral geometry on orbifolds is the open question whether an orbifold with singular points can be isospectral to a manifold. Several obstructions have emerged so far.

Theorem 3.3.1 ([GR03] Prop. 3.4(ii))**.** *Let $\mathcal{O}$ be a good compact Riemannian orbifold with singularities and let M be a compact Riemannian manifold such that $\mathcal{O}$ and M have a common Riemannian covering. Then M and $\mathcal{O}$ are not isospectral.*

Since having a common Riemannian covering (in the sense of orbifold coverings, cf. [Thu81]) is equivalent to the existence of a Riemannian manifold N and groups Γ_1, Γ_2 acting properly discontinuously and isometrically on $\mathcal{O}, N$, respectively, such that $\mathcal{O} \simeq N/\Gamma_1$, $M \simeq N/\Gamma_2$, this theorem immediately implies that Theorems 3.2.1 and 3.2.2 cannot yield isospectral examples of singular orbifolds and manifolds. This is clear anyway, however, because if two isometries of a manifold are conjugate, then one has fixed points if and only if the other one has. However, 3.3.1 also applies to isospectral flat orbifolds (which can be found, e.g., by using [MR03]), since isospectral pairs in this class are always covered by the same Euclidean space.

A similiar argument as in the proof of the theorem above yields the following obstruction. Note that this is no generalization of the theorem above, since the common covering in the theorem above is not assumed to be compact.

Theorem 3.3.2. *Let M_1 and M_2 be isospectral compact Riemannian manifolds and let Γ_1 and Γ_2 be finite groups of isometries acting effectively on M_1 and M_2, respectively. If M_1/Γ_1 and M_2/Γ_2 are isospectral orbifolds, then either both are manifolds or both are singular.*

Proof. First let $\mathcal{O}$ be a compact Riemannian orbifold with eigenvalues $0 = \mu_0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \ldots$. We will use a result from [DGGW08]: Let $S(\mathcal{O})$ denote the (finite) set of singular strata on $\mathcal{O}$, i.e., a certain partition of the singular set in $\mathcal{O}$ into submanifolds for which the isotropy on each element of $S(\mathcal{O})$ is constant. Moreover, dvol_N will denote the Riemannian density on an element N of $S(\mathcal{O})$. Then according to [DGGW08] Thm. 4.8 the heat trace $\sum_{j=0}^{\infty} e^{-\mu_j t}$ of $\mathcal{O}$ is for $t \searrow 0$ asymptotically equivalent to

$$I_0(\mathcal{O}) + \sum_{N \in S(\mathcal{O})} \frac{I_N}{|\operatorname{Iso}(N)|}, \tag{3.4}$$

where

$$\begin{aligned} I_0(\mathcal{O}) &= (4\pi t)^{-\dim(\mathcal{O})/2} \sum_{k=0}^{\infty} a_k(\mathcal{O}) t^k, \\ I_N &= (4\pi t)^{-\dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N b_k(N,x)\, \mathrm{dvol}_N(x) \end{aligned}$$

(as formal power series) for certain coefficients $a_k(\mathcal{O}) \in \mathbb{R}$, $b(N, \cdot) \in C^\infty(N)$, which satisfy

(i) $a_0(\mathcal{O}) = \mathrm{vol}(\mathcal{O})$

(ii) $\forall N \in S(\mathcal{O}), x \in N\colon\ b_0(N,x) > 0$

(iii) If Γ is a finite group of isometries acting an a compact Riemannian manifold M then $\forall k\colon\ a_k(M/\Gamma) = \frac{1}{|\Gamma|} a_k(M)$.

Now let M_1, M_2 be isospectral manifolds and let Γ_i act on M_i such that M_1/Γ_1 and M_2/Γ_2 are isospectral. Then (i) and the two isospectrality relations imply that $|\Gamma_1| =$

$|\Gamma_2|$. From this, the isospectrality of M_1 and M_2 and (iii) we deduce that $I_0(M_1/\Gamma_1) = I_0(M_2/\Gamma_2)$. Finally, (ii), (3.4) and the isospectrality of M_1/Γ_1 and M_2/Γ_2 imply that $S(M_1/\Gamma_1)$ is empty if and only if $S(M_2/\Gamma_2)$ is empty, i.e., M_1/Γ_1 is nonsingular if and only if M_2/Γ_2 is. □

Note that the theorem above (but not Theorem 3.3.1) applies to the isospectral orbifolds from Theorem 3.2.4.

These two obstructions are a prime motivation for the study of bad orbifolds. However, in this situation at least the following obstruction applies.

Theorem 3.3.3 ([DGGW08] Thm. 5.1)**.** *Let $\mathcal{O}$ be a compact Riemannian orbifold with singularities. If $\mathcal{O}$ is even-dimensional (respectively, odd-dimensional) and some $\mathcal{O}$-stratum of the singular set is odd-dimensional (respectively, even-dimensional), then $\mathcal{O}$ cannot be isospectral to a Riemannian manifold.*

Note that it would be pointless to apply these obstructions to our examples of isospectral bad orbifolds in Chapter 5, since the isospectral pairs given there are always diffeomorphic by definition. For more results on the spectral geometry of orbifolds see [Far01], [DGGW08] and the references therein.

4 The Torus Method for Orbifolds

In this chapter we present generalizations of the results in [Sch01] to orbifolds. Note that from this chapter on we use a slightly different notation than for the quotient orbifolds in Chapter 2: A point in a manifold M will usually be denoted by x (and not by $\tilde{x}$ as before).

4.1 Isospectral Metrics

Let G be a compact Lie group acting effectively on a connected manifold M. Recall the definition of the generic stabilizer $[H]$ in Proposition 3.2.3. If G is abelian, (i) and (ii) from that proposition imply that $H = \{e\}$ and that

$$M_G := \{x \in M;\ G_x = \{e\}\}$$

is open and dense in M.

Now let T be a torus (i.e. a nontrivial compact connected abelian Lie group) acting effectively and smoothly on a connected orbifold $\mathcal{O}$. Recall from Remark 2.1.12 that $\mathcal{O}^{\mathrm{reg}}$ is connected, open and dense in $\mathcal{O}$. It is also T-invariant, because the elements of T are diffeomorphisms and hence leave $\mathcal{O}^{\mathrm{reg}}$ invariant. From the preceding paragraph we conclude that the (not necessarily connected) manifold $\mathcal{O}^{\mathrm{reg}}_T := (\mathcal{O}^{\mathrm{reg}})_T$ is open and dense in $\mathcal{O}$.

Let $\mathfrak{t} = T_eT$ denote the Lie algebra of T. Setting $\mathcal{L} = \ker(\exp\colon \mathfrak{t} \to T)$, we observe that exp induces an isomorphism from $\mathfrak{t}/\mathcal{L}$ to T. Let $\mathcal{L}^* := \{\phi \in \mathfrak{t}^*;\ \phi(X) \in \mathbb{Z}\ \forall X \in \mathcal{L}\}$ denote the dual lattice. For an orbifold metric g on $\mathcal{O}$ we also write g for the induced (manifold) metric on $\mathcal{O}^{\mathrm{reg}}$ and its submanifolds. Using this and the notation from Corollary 2.3.11 for the submersion metric one has the following theorem. Note that we do not assume $\mathcal{O}$ to be oriented and dvol_g stands for the Riemannian density on $(\mathcal{O}, g)$.

Theorem 4.1.1. *Let T be a torus acting effectively and isometrically on two compact connected Riemannian orbifolds $(\mathcal{O}, g)$ and $(\mathcal{O}', g')$. Set $\widehat{\mathcal{O}} = \mathcal{O}^{reg}_T$, $\widehat{\mathcal{O}}' = \mathcal{O}'^{reg}_T$. Assume that for every subtorus $W \subset T$ of codimension 1 there is a T-equivariant diffeomorphism $F_W\colon \mathcal{O} \to \mathcal{O}'$ satisfying $F_W^* \mathrm{dvol}_{g'} = \mathrm{dvol}_g$ which induces an isometry between the manifolds $(\widehat{\mathcal{O}}/W, g^W)$ and $(\widehat{\mathcal{O}}'/W, g'^W)$. Then the orbifolds $(\mathcal{O}, g)$ and $(\mathcal{O}', g')$ are isospectral.*

Proof. We simply follow the proof for the manifold case in [Sch01] Theorem 1.4: Consider the Sobolev spaces $H := H^1(\mathcal{O}, g)$ and $H' := H^1(\mathcal{O}', g')$. We shall construct an isometry

$H' \to H$ preserving L^2-norms. Consider the following unitary representation of T on H: For $z \in T$, $f \in H$, $x \in \mathcal{O}$ set

$$(zf)(x) := f(zx).$$

Recall that all irreducible unitary representations of T are one-dimensional and given by homomorphisms $T \to S^1$. The latter have the form

$$T \simeq \mathfrak{t}/\mathcal{L} \ni [Z] \mapsto e^{2\pi i \mu(Z)} \in S^1$$

with μ running over $\mathcal{L}^*$ (cf. [BtD95] II.8). Hence the isotypic decomposition (cf. [Sep07] Thm. 3.19) of the T-module H is given by the Hilbert space direct sum

$$H = \bigoplus_{\mu \in \mathcal{L}^*} H_\mu$$

of T-modules $H_\mu = \{f \in H;\ [Z]f = e^{2\pi i \mu(Z)} f\ \forall Z \in \mathfrak{t}\}$. Note that H_0 is just the space of T-invariant functions in H.

For each subtorus W of T of codimension 1 set

$$E_W := \bigoplus_{\substack{\mu \in \mathcal{L}^* \setminus \{0\} \\ T_e W = \ker \mu}} H_\mu$$

and denote the (Hilbert) sum over all these subtori by $\bigoplus_W$. We obtain the decomposition

$$H = H_0 \oplus \bigoplus_{\mu \in \mathcal{L}^* \setminus \{0\}} H_\mu = H_0 \oplus \bigoplus_W E_W.$$

Moreover, set

$$H_W := H_0 \oplus E_W = \bigoplus_{\substack{\mu \in \mathcal{L}^* \\ T_e W \subset \ker \mu}} H_\mu$$

and note that H_W consists precisely of the W-invariant functions in H: Obviously all functions in H_W are W-invariant. Conversely, let $f \in H$ be W-invariant. If W' is another subtorus of T of codimension 1, then the projection $f^{W'}$ of f into $E_{W'}$ is zero since $f^{W'}$ is invariant by W and W' and hence $f^{W'} \in H_0$. Therefore $f \in H_0 \oplus E_W = H_W$.

Now use the analogous notation H'_μ, E'_W, H'_W for the corresponding subspaces of H'. Fix a subtorus W of T of codimension 1 and let $F_W \colon \mathcal{O} \to \mathcal{O}'$ be the corresponding diffeomorphism from the assumption. Since F_W is T-equivariant, F_W^* maps H'_0 to H_0 and H'_W to H_W. We will show that $F_W^* \colon H'_W \to H_W$ is a Hilbert space isometry preserving the L^2-norm. It obviously preserves the L^2-norm because of $F_W^* \,\mathrm{dvol}_{g'} = \mathrm{dvol}_g$ and Lemma 2.4.4.

Let $\psi \in C^\infty(\mathcal{O}')$ be invariant under W and let $y \in \widehat{\mathcal{O}}'$. Set

$$\phi = F_W^* \psi,\ x := F_W^{-1}(y) \in \widehat{\mathcal{O}}.$$

Note that $\phi \in C^\infty(\mathcal{O})$ is also W-invariant and let $\overline{\phi}$, $\overline{\psi}$ denote the induced functions on

$\widehat{\mathcal{O}}/W$ and $\widehat{\mathcal{O}}'/W$, respectively. Since $\operatorname{grad}\phi$ and $\operatorname{grad}\psi$ are W-horizontal vector fields on $\hat{\mathcal{O}}$, $\hat{\mathcal{O}}'$, respectively, and the map $\overline{F}_W\colon (\widehat{\mathcal{O}}/W, g^W) \to (\widehat{\mathcal{O}}'/W, g'^W)$ induced by F_W is an isometry, we have, by the definition of the Riemannian submersion metric,

$$\|\operatorname{grad}\phi(x)\|_g = \|\operatorname{grad}\overline{\phi}(\overline{x})\|_{g^W} = \|\operatorname{grad}\overline{\psi}(\overline{y})\|_{g'^W} = \|\operatorname{grad}\psi(y)\|_{g'},$$

where $\overline{x} \in \widehat{\mathcal{O}}/W$ and $\overline{y} \in \widehat{\mathcal{O}}'/W$ denote the images of x, respectively y, under the canonical projection. Since $\widehat{\mathcal{O}}$ is dense in $\mathcal{O}$ and $\widehat{\mathcal{O}'}$ is dense in $\mathcal{O}'$, this implies that $F_W^*\colon H'_W \to H_W$ is a Hilbert space isometry with respect to the H^1-product. Since the map $F_W^*\colon H'_W \to H_W$ is a Hilbert space isometry preserving L^2-norms, so is its restriction $F^*_{W|E'_W}\colon E'_W \to E_W$.

But these maps for all subtori $W \subset T$ of codimension 1 give an isometry from $\bigoplus_W E'_W$ to $\bigoplus_W E_W$ preserving L^2-norms. Choosing an isometry $H'_0 \to H_0$ given by an arbitrary F_W^*, we obtain an L^2-norm-preserving isometry $H' \to H$. Isospectrality of $(\mathcal{O}, g)$ and $(\mathcal{O}', g')$ now follows from Theorem 3.1.4. □

Remark. If $\mathcal{O}$ were an oriented orbifold and dvol_g above denoted the Riemannian volume form, we would have to rephrase the theorem assuming that F_W is orientation-preserving and apply Lemma 2.4.1 instead of 2.4.4 in the proof.

We now fix a torus T and use the notation $\mathfrak{t} = T_eT$, $\mathcal{L} = \ker(\exp\colon \mathfrak{t} \to T)$ as above. Moreover, we fix a compact connected Riemannian orbifold $(\mathcal{O}, g_0)$ and a smooth effective action of T on $(\mathcal{O}, g_0)$ by isometries and set $\widehat{\mathcal{O}} := \mathcal{O}_T^{\mathrm{reg}}$. For $Z \in \mathfrak{t}$ we write $Z\widehat{\ } := \widehat{Z} := Z^*_{\mathrm{reg}|\widehat{\mathcal{O}}}$ for the fundamental vector field on $\widehat{\mathcal{O}}$ induced by Z. (For the notation Z^*_{reg} recall Remark 2.3.5.) We will need the following definitions and results, which generalize [Sch01] 1.5 to our orbifold setting.

1. A $\mathfrak{t}$-valued 1-form on $\mathcal{O}$ will be called *admissible* if it is T-invariant and T-horizontal in the sense of Definition 2.3.19.

2. For an admissible 1-form λ on the orbifold $\mathcal{O}$ denote by g_λ the Riemannian metric on $\mathcal{O}$ given in a chart $(U, \tilde{U}/\Gamma, \pi)$ on $\mathcal{O}$ by

   $$g_{\lambda\pi}(X, Y) = g_{0\pi}(X + (\lambda_\pi(X))^*_\pi, Y + (\lambda_\pi(Y))^*_\pi)$$

   for $X, Y \in \mathcal{V}(\tilde{U})$. To check that g_λ is indeed an orbifold tensor field, it suffices to show that for charts $(U_i, \tilde{U}_i/\Gamma_i, \pi), i = 1, 2$, on $\mathcal{O}$ with $U_1 \subset U_2$ and an injection μ from π_1 to π_2, we have $g_{\lambda,\pi_1} = \mu^* g_{\lambda,\pi_2}$. But this follows from the analogous compatibility conditions for λ, fundamental vector fields Z^* and for g_0: For $\tilde{x} \in \tilde{U}_1$

and $X, Y \in T_{\tilde{x}}\tilde{U}_1$ we calculate

$$\begin{aligned}
\mu^* g_{\lambda,\pi_2}(X,Y) &= g_{\lambda,\pi_2}(\mu_* X, \mu_* Y) \\
&= g_{0,\pi_2}(\mu_* X + (\lambda_{\pi_2}(\mu_* X))^*_{\pi_2}(\mu(\tilde{x})), \mu_* Y + (\lambda_{\pi_2}(\mu_* Y))^*_{\pi_2}(\mu(\tilde{x}))) \\
&= g_{0,\pi_2}(\mu_* X + (\lambda_{\pi_1}(X))^*_{\pi_2}(\mu(\tilde{x})), \mu_* Y + (\lambda_{\pi_1}(Y))^*_{\pi_2}(\mu(\tilde{x}))) \\
&= g_{0,\pi_2}(\mu_* X + \mu_*(\lambda_{\pi_1}(X))^*_{\pi_1}(\tilde{x}), \mu_* Y + \mu_*(\lambda_{\pi_1}(Y))^*_{\pi_1}(\tilde{x})) \\
&= g_{0,\pi_1}(X + (\lambda_{\pi_1}(X))^*_{\pi_1}(\tilde{x}), Y + (\lambda_{\pi_1}(Y))^*_{\pi_1}(\tilde{x})) \\
&= g_{\lambda,\pi_1}(X,Y).
\end{aligned}$$

Note that if $\Phi_{\lambda,\pi}$ denotes the $C^\infty(\tilde{U})$-isomorphism

$$\mathcal{V}(\tilde{U}) \ni X \mapsto X - (\lambda_\pi(X))^*_\pi \in \mathcal{V}(\tilde{U}),$$

then $g_{\lambda\pi} = (\Phi^{-1}_{\lambda,\pi})^* g_{0\pi}$. Since λ is horizontal, $\Phi_{\lambda,\pi}$ is unipotent and this implies

$$\mathrm{dvol}_{g_{\lambda,\pi}} = |\det \Phi^{-1}_{\lambda,\pi}|\, \mathrm{dvol}_{g_{0,\pi}} = \mathrm{dvol}_{g_{0,\pi}}\,.$$

Since this holds for every chart π, we have $\mathrm{dvol}_{g_\lambda} = \mathrm{dvol}_{g_0}$.

3. Since λ and g_0 are T-invariant and T is abelian, we can use Lemma 2.3.20 to show that g_λ is T-invariant: If $z \in T$ and $(U_i, \tilde{U}_i/\Gamma_i, \pi_i), i = 1, 2,$ are charts on $\mathcal{O}$ such that there is a diffeomorphism $\tilde{z} \in C^\infty(\tilde{U}_1, \tilde{U}_2)$ satisfying $\pi_2 \circ \tilde{z} = z \circ \pi_1$, then for $\tilde{x} \in \tilde{U}_1$, $X, Y \in T_{\tilde{x}}\tilde{U}_1$ we have

$$\tilde{z}_*(\lambda_1(X))^*_{\pi_1}(\tilde{x}) = (\lambda_1(X))^*_{\pi_2}(\tilde{z}(\tilde{x})) = (\lambda_2(\tilde{z}_* X))^*_{\pi_2}(\tilde{z}(\tilde{x}))$$

(and analogously for Y) and hence

$$\begin{aligned}
\tilde{z}^* g_{\lambda,\pi_2}(X,Y) &= g_{\lambda,\pi_2}(\tilde{z}_* X, \tilde{z}_* Y) \\
&= g_{0,\pi_2}\left(\tilde{z}_* X + (\lambda_{\pi_2}(\tilde{z}_* X))^*_{\pi_2}(\tilde{z}(\tilde{x})), \tilde{z}_* Y + (\lambda_{\pi_2}(\tilde{z}_* Y))^*_{\pi_2}(\tilde{z}(\tilde{x}))\right) \\
&= g_{0,\pi_2}\left(\tilde{z}_* X + \tilde{z}_*(\lambda_{\pi_1}(X))^*_{\pi_1}(\tilde{x}), \tilde{z}_* Y + \tilde{z}_*(\lambda_{\pi_1}(Y))^*_{\pi_1}(\tilde{x})\right) \\
&= g_{0,\pi_1}\left(X + (\lambda_{\pi_1}(X))^*_{\pi_1}(\tilde{x}), Y + (\lambda_{\pi_1}(Y))^*_{\pi_1}(\tilde{x})\right) \\
&= g_{\lambda,\pi_1}(X,Y).
\end{aligned}$$

4. Moreover, note that for every $x \in \widehat{\mathcal{O}}$ the metric g_λ on $T_x\widehat{\mathcal{O}}$ restricts to the same metric as g_0 on the vertical subspace $\mathfrak{t}_x = \{\hat{Z}_x; Z \in \mathfrak{t}\} \subset T_x\widehat{\mathcal{O}}$, because λ is T-horizontal. Since for every $X \in \mathcal{V}(\widehat{\mathcal{O}})$ the vector field $(\widehat{\lambda}(X))\widehat{\ }$ is by definition vertical (where $\widehat{\lambda}$ denotes the restriction of λ_{reg} to $\widehat{\mathcal{O}}$), the metrics g_0^T and g_λ^T on $\widehat{\mathcal{O}}/T$ coincide.

With λ an admissible 1-form on $\mathcal{O}$ as above we set $\widehat{\Phi}_\lambda := \Phi_{\lambda,\mathrm{id}_{\widehat{\mathcal{O}}}}\colon \mathcal{V}(\widehat{\mathcal{O}}) \to \mathcal{V}(\widehat{\mathcal{O}})$. In the proof of the following theorem we follow [Sch01] Thm. 1.6.

Theorem 4.1.2. *Let λ, λ' be two admissible 1-forms on $\mathcal{O}$ satisfying:*
For every $\mu \in \mathcal{L}^*$ there is a T-equivariant $F_\mu \in \operatorname{Isom}(\mathcal{O}, g_0)$ such that

$$\mu \circ \lambda = F_\mu^*(\mu \circ \lambda'). \tag{4.1}$$

Then $(\mathcal{O}, g_\lambda)$ and $(\mathcal{O}, g_{\lambda'})$ are isospectral.

Proof. We shall use Theorem 4.1.1. So let W be a subtorus of T of codimension 1 and choose $\mu \in \mathcal{L}^*$ such that $\ker \mu = T_e W$. Let $F_\mu \in \operatorname{Isom}(\mathcal{O}, g_0)$ be a corresponding isometry satisfying (4.1). We will show that $F_W := F_\mu$ satisfies the conditions of Theorem 4.1.1. Since F_μ is an isometry, we have by the remarks above

$$F_\mu^* \operatorname{dvol}_{g_{\lambda'}} = F_\mu^* \operatorname{dvol}_{g_0} = \operatorname{dvol}_{g_0} = \operatorname{dvol}_{g_\lambda}.$$

To see that F_μ induces an isometry between the manifolds $(\widehat{\mathcal{O}}/W, g_\lambda^W)$ and $(\widehat{\mathcal{O}}/W, g_{\lambda'}^W)$, let $x \in \widehat{\mathcal{O}}$ and let $V \in T_x\widehat{\mathcal{O}}$ be W-horizontal with respect to g_λ. Set $X := \widehat{\Phi}_\lambda^{-1}(V) \in T_x\widehat{\mathcal{O}}$ and note that (4.1) implies that

$$Z := \widehat{\lambda}'(F_{\mu_*}X) - \widehat{\lambda}(X) \in \ker \mu.$$

Set $Y := \widehat{\Phi}_{\lambda'}(F_{\mu_*}X)$ and observe that $F_{\mu_*}V - Y \in T_{F_\mu(x)}\widehat{\mathcal{O}}$ is W-vertical:

$$\begin{aligned}
F_{\mu_*}V - Y &= F_{\mu_*}(\widehat{\Phi}_\lambda X) - \widehat{\Phi}_{\lambda'}(F_{\mu_*}X) \\
&= F_{\mu_*}(X - \widehat{\widehat{\lambda}(X)}_x) - F_{\mu_*}X + \widehat{\widehat{\lambda}'(F_{\mu_*}X)}_{F_\mu(x)} \\
&= \widehat{\widehat{\lambda}'(F_{\mu_*}X)}_{F_\mu(x)} - F_{\mu_*}(\widehat{\widehat{\lambda}(X)}_x) \\
&= \widehat{\widehat{\lambda}'(F_{\mu_*}X)}_{F_\mu(x)} - \widehat{\widehat{\lambda}(X)}_{F_\mu(x)} = \widehat{Z}_{F_\mu(x)}
\end{aligned}$$

where we used that F_μ is T-equivariant.

Moreover, note that Y is W-horizontal with respect to $g_{\lambda'}$: Since λ is T-horizontal and V is W-horizontal with respect to g_λ, the vector $X = \widehat{\Phi}_\lambda^{-1}(V) \in T_x\widehat{\mathcal{O}}$ is W-horizontal with respect to g_0. Hence so is $F_{\mu_*}X$, since F_μ is T-equivariant and a g_0-isometry.

These two observations imply that Y is the $g_{\lambda'}$-horizontal part of $F_{\mu_*}V$. Since

$$\|Y\|_{g_{\lambda'}} = \|F_{\mu_*}X\|_{g_0} = \|X\|_{g_0} = \|V\|_{g_\lambda}$$

and the W-horizontal vector V was chosen arbitrarily, we conclude that F_μ indeed induces an isometry between $(\widehat{\mathcal{O}}/W, g_\lambda^W)$ and $(\widehat{\mathcal{O}}/W, g_{\lambda'}^W)$.

Finally, the isospectrality of $(\mathcal{O}, g_\lambda)$ and $(\mathcal{O}, g_{\lambda'})$ follows from Theorem 4.1.1. □

4.2 Nonisometry

In this section we give a sufficient criterion under which two orbifolds $(\mathcal{O}, g_\lambda)$ and $(\mathcal{O}, g_{\lambda'})$ as in Theorem 4.1.2 are not isometric. Let $(\mathcal{O}, g_0)$, T, $\mathfrak{t}$, $\mathcal{L}$, $\widehat{\mathcal{O}}$ be as in the preceding section. Note that the action of T on $\widehat{\mathcal{O}}$ gives $\widehat{\mathcal{O}}$ the structure of a principal T-bundle $\pi\colon \widehat{\mathcal{O}} \to \widehat{\mathcal{O}}/T$. By λ, λ' we will always denote admissible $\mathfrak{t}$-valued 1-forms on $\mathcal{O}$. The following notations and the resulting lemma are just minor generalizations of [Sch01] 2.1 and 2.2.

Notations and Remarks 4.2.1. (i) A diffeomorphism $F\colon \mathcal{O} \to \mathcal{O}$ is called *T-preserving* if conjugation by F preserves $T \subset \mathrm{Diffeo}(\mathcal{O})$, i.e. $c^F(z) := F \circ z \circ F^{-1} \in T \ \forall z \in T$. In this case we denote by $\Psi_F := c^F_*$ the automorphism of $\mathfrak{t} = T_eT$ induced by the isomorphism c^F on T. Obviously, each T-preserving diffeomorphism F of $\mathcal{O}$ maps T-orbits to T-orbits. In particular, F preserves not only $\mathcal{O}^{\mathrm{reg}}$, but also $\widehat{\mathcal{O}}$. Recall that for $Z \in \mathfrak{t}$ we denote by $\widehat{Z}$ the fundamental vector field on $\widehat{\mathcal{O}}$ associated with Z. Then for $x \in \widehat{\mathcal{O}}$:

$$\begin{aligned} F_*\widehat{Z}_x &= F_*\left(\frac{d}{dt}_{|t=0}(\exp(tZ)x)\right) = \frac{d}{dt}_{|t=0}(F(\exp(tZ)x)) \\ &= \frac{d}{dt}_{|t=0}(F \circ \exp(tZ) \circ F^{-1})(F(x)) = \frac{d}{dt}_{|t=0}(c^F(\exp(tZ))(F(x)) \\ &= \frac{d}{dt}_{|t=0}(\exp(\Psi_F(tZ))(F(x))) = \widehat{\Psi_F(Z)}_{F(x)}. \end{aligned}$$

Hence, we have $F_*\widehat{Z} = \widehat{\Psi_F(Z)}$.

(ii) We denote by $\mathrm{Aut}^T_{g_0}(\mathcal{O})$ the group of all T-preserving diffeomorphisms of $\mathcal{O}$ which, in addition, preserve the g_0-norm of vectors tangent to the T-orbits in $\widehat{\mathcal{O}}$ and induce an isometry of the Riemannian manifold $(\widehat{\mathcal{O}}/T, g_0^T)$. We denote the corresponding group of induced isometries by $\overline{\mathrm{Aut}}^T_{g_0}(\mathcal{O}) \subset \mathrm{Isom}(\widehat{\mathcal{O}}/T, g_0^T)$.

(iii) We define $\mathcal{D} := \{\Psi_F;\ F \in \mathrm{Aut}^T_{g_0}(\mathcal{O})\} \subset \mathrm{Aut}(\mathfrak{t})$. Note that $\mathcal{D}$ is discrete because it is a subgroup of the discrete group $\{\Psi \in \mathrm{Aut}(\mathfrak{t});\ \Psi(\mathcal{L}) = \mathcal{L}\}$.

(iv) Let $\omega_0\colon T\widehat{\mathcal{O}} \to \mathfrak{t}$ denote the connection form on the principal T-bundle $\widehat{\mathcal{O}}$ associated with g_0; i.e. $\omega_0(\widehat{Z}) = Z \ \forall Z \in \mathfrak{t}$ and for each $x \in \widehat{\mathcal{O}}$ the kernel $\ker(\omega_{0|T_x\widehat{\mathcal{O}}})$ is the g_0-orthogonal complement of the vertical space $\mathfrak{t}_x = \{\widehat{Z}_x;\ Z \in \mathfrak{t}\}$ in $T_x\widehat{\mathcal{O}}$.

The connection form on $\widehat{\mathcal{O}}$ associated with g_λ is given by $\omega_\lambda := \omega_0 + \hat{\lambda}$: Since λ is horizontal, $\omega_\lambda(\widehat{Z}) = \omega_0(\widehat{Z}) + \hat{\lambda}(\widehat{Z}) = \omega_0(\widehat{Z}) = Z$. Morover, for each $x \in \widehat{\mathcal{O}}$ the kernel $\ker(\omega_{\lambda|T_x\widehat{\mathcal{O}}})$ is indeed the g_λ-orthogonal complement of t_x: For $X \in T_x\widehat{\mathcal{O}}$ we have the equivalence

$$\begin{aligned} 0 = \omega_\lambda(X) = \omega_0(X) + \hat{\lambda}(X) = \omega_0(X + \widehat{(\hat{\lambda}(X))}_x) \\ \iff \forall Z \in \mathfrak{t}\colon\ 0 = g_0(X + \widehat{(\hat{\lambda}(X))}_x, \widehat{Z}_x) = g_\lambda(X, \widehat{Z}_x). \end{aligned}$$

(v) Let Ω_λ denote the curvature form on the manifold $\widehat{\mathcal{O}}/T$ associated with the connection form ω_λ on $\widehat{\mathcal{O}}$. We have

$$\pi^*\Omega_\lambda = d\omega_\lambda + \frac{1}{2}[\omega_\lambda, \omega_\lambda] = d\omega_\lambda$$

because T is abelian.

(vi) Since $\widehat{\lambda}$ is T-invariant and T-horizontal, it induces some $\mathfrak{t}$-valued 1-form $\overline{\lambda}$ on $\widehat{\mathcal{O}}/T$. Then $\pi^*\Omega_\lambda = d\omega_\lambda = d\omega_0 + d\widehat{\lambda}$ implies

$$\Omega_\lambda = \Omega_0 + d\overline{\lambda}.$$

Lemma 4.2.2. *Let $F\colon (\mathcal{O}, g_\lambda) \to (\mathcal{O}, g_{\lambda'})$ be a T-preserving isometry. Then:*

(i) F preserves the g_0-norm of vectors tangent to the T-orbits in $\widehat{\mathcal{O}}$, and it induces an isometry $\bar{F}$ of $(\widehat{\mathcal{O}}/T, g_0^T)$. In particular, $F \in \mathrm{Aut}^T_{g_0}(\mathcal{O})$ and $\Psi_F \in \mathcal{D}$.

(ii) $F^*\omega_{\lambda'} = \Psi_F \circ \omega_\lambda \in \Omega^1(\widehat{\mathcal{O}}, \mathfrak{t})$, in particular $F^*d\omega_{\lambda'} = \Psi_F \circ d\omega_\lambda$.

(iii) The isometry $\bar{F}$ of $(\widehat{\mathcal{O}}/T, g_0^T)$ satisfies $\bar{F}^*\Omega_{\lambda'} = \Psi_F \circ \Omega_\lambda$.

Proof. (i) The first statement holds since $g_\lambda = g_0 = g_{\lambda'}$ on vertical spaces in $\widehat{\mathcal{O}}$; recall Remark 4 in Section 4.1. F induces an isometry $\bar{F}$ because $g^T_\lambda = g^T_0 = g^T_{\lambda'}$ on $\widehat{\mathcal{O}}/T$.

(ii) First we show that both forms coincide on vertical spaces: As above, for $Z \in \mathfrak{t}$ write $\hat{Z} := Z^*_{\mathrm{reg}|\widehat{\mathcal{O}}}$. Using that $\omega_\lambda, \omega_{\lambda'}$ are connection forms and 4.2.1 (i) we obtain

$$\omega_{\lambda'}(F_*(\hat{Z})) = \omega_{\lambda'}(\widehat{\Psi_F(Z)}) = \Psi_F(Z) = \Psi_F(\omega_\lambda(\hat{Z})).$$

Thus the equation holds on vertical spaces. It also holds on g_λ-horizontal vectors: Note that the right hand side vanishes on g_λ-horizontal vectors by definition. Since F maps orbits to orbits, its differential maps vertical to vertical vectors. Since it is also an isometry, F_* maps g_λ-horizontal vectors to $g_{\lambda'}$-horizontal vectors. Hence the left-hand side, too, vanishes on g_λ-horizontal vectors.

(iii) This follows from (ii) and 4.2.1 (v).

□

We will later need the following lemma, for which we have not been able to find a reference.

Lemma 4.2.3. *Let G be a connected abelian Lie group and let $P \to M$ be a connected G-principal fibre bundle. Moreover, let $\omega\colon TP \to \mathfrak{g}$ be a connection form on P. If $F\colon P \to P$ is a gauge transformation which preserves ω, then $F \in G$.*

Proof. Write $F(p) = \Phi(p)p$ for $p \in P$ with a smooth function $\Phi\colon P \to G$. We will show that Φ is constant on P. Let $c\colon (-\varepsilon, \varepsilon) \to P$ be a curve in P and write

$$\phi(t) := \Phi(c(t)) \in G.$$

To show that Φ is constant, it suffices to show $\dot{\phi}(0) = 0$. Now denote by $\tilde{\phi}$ the curve $(-\varepsilon, \varepsilon) \ni t \mapsto \phi(0)^{-1}\phi(t) \in G$ and consider the smooth map

$$h\colon\ (-\varepsilon, \varepsilon)^2 \ni (s,t) \mapsto \tilde{\phi}(s)c(t) \in P.$$

Then

$$\begin{aligned} F_*\dot{c}(0) = (F \circ c)^{\cdot}(0) &= \phi(0)_* \left[\frac{d}{dt}_{|t=0} h(t,t)\right] = \phi(0)_* \left[\frac{d}{dt}_{|t=0} h(0,t) + \frac{d}{dt}_{|t=0} h(t,0)\right] \\ &= \phi(0)_* \left[\tilde{\phi}(0)_*\dot{c}(0) + (\dot{\tilde{\phi}}(0))^*_{c(0)}\right] = \phi(0)_* \left[\dot{c}(0) + (\dot{\tilde{\phi}}(0))^*_{c(0)}\right], \end{aligned} \tag{4.2}$$

where we have used that for any $p \in P$ the map $G \ni g \mapsto gp \in P$ is smooth and hence applying its differential to $\frac{d}{dt}_{|t=0}\tilde{\phi}(t) = \frac{d}{dt}_{|t=0} \exp(t\dot{\tilde{\phi}}(0))$ gives

$$\frac{d}{dt}_{|t=0}\tilde{\phi}(t)p = \frac{d}{dt}_{|t=0} \exp(t\dot{\tilde{\phi}}(0))p = (\dot{\tilde{\phi}}(0))^*_p.$$

Since ω is invariant under F and under $\phi(0)$ (because G is abelian), applying ω to (4.2) gives

$$\omega(\dot{c}(0)) = \omega(\dot{c}(0)) + \dot{\tilde{\phi}}(0);$$

hence, $\dot{\phi}(0) = \phi(0)_*\dot{\tilde{\phi}}(0) = 0$ as claimed. □

Before coming to the following propositions, note that the isometry group $\mathrm{Isom}(\mathcal{O}, g)$ of a Riemannian orbifold $(\mathcal{O}, g)$ endowed with the compact-open topology admits a unique smooth structure that turns it into a Lie group ([BZ07]). The proof of the following proposition is analogous to the proof of [Sch01] Prop. 2.3.

Proposition 4.2.4. *Let λ be an admissible $\mathfrak{t}$-valued 1-form on $\mathcal{O}$ such that the associated curvature form Ω_λ on $\widehat{\mathcal{O}}/T$ satisfies the following genericity condition:*

(G) No nontrivial 1-parameter group in $\overline{Aut}^T_{g_0}(\mathcal{O})$ preserves Ω_λ.

Then T is a maximal torus in $\mathrm{Isom}(\mathcal{O}, g_\lambda)$

Proof. Assume that $F_t \in \mathrm{Isom}(\mathcal{O}, g_\lambda)$ is a 1-parameter family of isometries commuting with T. If we can show that $F_t \in T\ \forall t$, we know that T is maximal. Since the F_t commute with T, they are T-preserving. By Lemma 4.2.2(i) they induce a 1-parameter family $\bar{F}_t \in \mathrm{Isom}(\widehat{\mathcal{O}}/T, g_0^T)$, hence $F_t \in \mathrm{Aut}^T_{g_0}(\mathcal{O})$ and $\Psi_{F_t} \in \mathcal{D}\ \forall t$. Since $\Psi_{F_0} = \Psi_{\mathrm{Id}} = \mathrm{Id}$ and $\mathcal{D}$ is discrete, we have $\Psi_{F_t} = \mathrm{Id}$ for all t and hence by Lemma 4.2.2(iii) each $\bar{F}_t$ preserves Ω_λ. By (G) this implies $\bar{F}_t = \mathrm{id}$ for all t. Hence each $F_{t|\widehat{\mathcal{O}}}$ is a gauge transformation of the principal bundle $\widehat{\mathcal{O}} \to \widehat{\mathcal{O}}/T$. But now Lemma 4.2.2(ii) and Lemma 4.2.3 imply that $F_{t|\widehat{\mathcal{O}}}$ acts as an element of T on every connected component of $\widehat{\mathcal{O}}$. Since the isometry $F_{t|\mathcal{O}^{\mathrm{reg}}}$ is determined uniquely by its values on an open set in the connected manifold $\mathcal{O}^{\mathrm{reg}}$ and $\mathcal{O}^{\mathrm{reg}}$ is dense in $\mathcal{O}$, we conclude that $F_t \in T$. □

Lemma 4.2.2 and the proposition above now imply the following proposition. Its proof is almost literally the same as that of [Sch01] Prop. 2.4 but we include it for completeness.

Proposition 4.2.5. *Let λ, λ' be admissible 1-forms on $\mathcal{O}$ such that $\Omega_{\lambda'}$ has property (G). Furthermore, assume that*

(N) $\Omega_\lambda \notin \mathcal{D} \circ \overline{\mathrm{Aut}}^T_{g_0}(\mathcal{O})^* \Omega_{\lambda'}$.

Then $(\mathcal{O}, g_\lambda)$ and $(\mathcal{O}, g_{\lambda'})$ are not isometric.

Proof. Suppose that there were an isometry $F\colon (\mathcal{O}, g_\lambda) \to (\mathcal{O}, g_{\lambda'})$. By Proposition 4.2.4, T is a maximal torus in $\mathrm{Isom}(\mathcal{O}, g_{\lambda'})$. Since $\{F \circ z \circ F^{-1};\ z \in T\}$ also is a torus in $\mathrm{Isom}(\mathcal{O}, g_{\lambda'})$ and all maximal tori are conjugate, we can assume F - after possibly combining it with an isometry of $(\mathcal{O}, g_{\lambda'})$ - to be T-preserving. But then Lemma 4.2.2 implies $\bar{F}^* \Omega_{\lambda'} = \Psi_F \circ \Omega_\lambda$ with $\bar{F} \in \overline{\mathrm{Aut}}^T_{g_0}(\mathcal{O})$ and $\Psi_F \in \mathcal{D}$, which contradicts our assumption. $\square$

5 Examples of Isospectral Bad Orbifolds

As mentioned in Chapter 3, one can easily obtain examples of isospectral bad orbifolds of the form $\mathcal{O} \times \mathcal{O}_1, \mathcal{O} \times \mathcal{O}_2$ from isospectral good orbifolds $\mathcal{O}_1, \mathcal{O}_2$ and a bad orbifold $\mathcal{O}$. However, in this chapter we will use the constructions from the preceding chapter to give genuinely new examples of isospectral bad orbifolds. More precisely, for every fixed $n \geqslant 4$ and coprime positive integers p, q we will give isospectral pairs and even families of metrics on certain $2n$-dimensional weighted projective spaces (depending on p, q). The latter turn out to be bad orbifolds for $(p, q) \neq (1, 1)$.

5.1 Weighted Projective Spaces

Consider the following orbifold which is a special weighted projective space: For $n \geqslant 4$ let $S^{2n+1} \subset \mathbb{C}^{n+1}$ denote the standard sphere and let p, q be coprime positive integers. Let $S^1 \subset \mathbb{C}$ act smoothly on S^{2n+1} by

$$\sigma(u, v) = (\sigma^p u, \sigma^q v), \tag{5.1}$$

where $\sigma \in S^1 \subset \mathbb{C}$, $u \in \mathbb{C}^{n-1}$, $v \in \mathbb{C}^2$.

We use Theorem 2.2.1 to show that the quotient under this action becomes an orbifold: Since $\gcd(p, q) = 1$, the action is free in all points $(u, v) \in S^{2n+1}$ with $u \neq 0$ and $v \neq 0$. In particular, S^1 acts effectively. Calculating the stabilizers of the other points, we see that the action is almost free: The points (u, v) with $v = 0$ are fixed precisely by the p-th roots of unity and the points of the form $(0, v) \in S^{2n+1}$ are fixed precisely by the q-th roots of unity. Since p and q are coprime, this implies that the fixed point set of a nontrivial p-th root of unity is an embedded S^{2n-3} and the fixed point set of a nontrivial q-th root of unity is diffeomorphic to S^3. Hence, each fixed point set has codimension at least $4 > \dim S^1 + 2$. By Theorem 2.2.1, the quotient $\mathcal{O} := \mathcal{O}(p, q) := S^{2n+1}/S^1$ under the action (5.1) indeed becomes an orbifold. Moreover, $\mathrm{Iso}([(u, 0)]) = \mathbb{Z}_p$ for $u \in S^{2n-3} \subset \mathbb{C}^{n-1}$, $\mathrm{Iso}([(0, v)]) = \mathbb{Z}_q$ for $v \in S^3 \subset \mathbb{C}^2$ and $\mathcal{O}^{\mathrm{reg}} = \{[(u, v)] \in \mathcal{O};\ u \neq 0 \wedge v \neq 0\}$.

For every pair (p, q) we will construct isospectral metrics on the orbifold $\mathcal{O} = \mathcal{O}(p, q)$. Note that for $p = q = 1$ we have $\mathcal{O} = \mathbb{CP}^n$. All other orbifolds in this family are singular.

Since S^{2n+1} is simply connected and S^1 is connected, the orbifold fundamental group $\pi_1^{\mathrm{Orb}}(\mathcal{O}(p, q))$ is trivial for all p, q ([ALR07] Proposition 1.54). This implies that each $\mathcal{O}(p, q)$ is its own universal covering orbifold (cf. [Thu81]) and hence that for $(p, q) \neq 1$ cannot be covered by any manifold. This means that the orbifolds $\mathcal{O}(p, q)$ for $(p, q) \neq$

$(1, 1)$ are “bad”, i.e., they cannot be written as a quotient of a manifold by a properly discontinuous group action.

Throughout this section, $\langle , \rangle$ will always denote the canonical metric on $S^{2n+1} \subset \mathbb{C}^{n+1}$ given by the restriction of the inner product

$$\langle X, Y\rangle = \mathrm{Re}\left(\sum_{i=1}^{n+1} X_i \bar{Y}_i\right) \text{ for } X, Y \in \mathbb{C}^{n+1}.$$

Besides, $\langle , \rangle$ will also denote the unique metric on $\mathcal{O} = S^{2n+1}/S^1$ with respect to which the quotient map $P\colon\ S^{2n+1} \to S^{2n+1}/S^1$ becomes a Riemannian orbifold submersion (cf. Corollary 2.3.11 and note that the above S^1-action is by isometries). In cases where the metric is not specified, we will always assume that $\langle , \rangle$ is used. The metric $\langle , \rangle$ on $\mathcal{O}$ will also be denoted by g_0.

Note that isospectral families of metrics on $\mathcal{O}(1, 1) = \mathbb{CP}^n$ have already been given in [Rüc06] using the manifold version of the construction in the following section. Similar methods have also led to examples of isospectral families of good orbifolds ([Sut06]). For results on the spectral geometry of weighted projectives spaces with their standard metric see [ADFG08] and [GUW08].

5.2 Isospectral Metrics

In this section we will give isospectral metrics on the orbifold $\mathcal{O} = \mathcal{O}(p, q)$. To this end we will use the torus method from the preceding chapter with $T = S^1 \times S^1 = \{(e^{is_1}, e^{is_2});\ s_j \in [0, 2\pi)\} \subset \mathbb{C}^2$. We identify $\mathbb{R}^2$ with $\mathfrak{t} = T_{(1,1)}(S^1 \times S^1)$ by

$$\mathbb{R}^2 \ni (t_1, t_2) \mapsto (it_1, it_2) \in \mathfrak{t} \subset \mathbb{C}^2$$

and set

$$Z_1 = (i, 0), Z_2 = (0, i) \in \mathfrak{t}.$$

We will need the following variation of [Sch01] Definition 3.2.4. (The only difference is a broader definition of equivalence in (ii).)

Definition 5.2.1. Let $j, j'\colon\ \mathfrak{t} \simeq \mathbb{R}^2 \to \mathfrak{su}(m)$ be two linear maps.

(i) We call j and j' *isospectral* if for each $Z \in \mathfrak{t}$ there is $A_Z \in SU(m)$ such that $j'_Z = A_Z j_Z A_Z^{-1}$.

(ii) Let $Q\colon\ \mathbb{C}^m \to \mathbb{C}^m$ denote complex conjugation and set

$$\mathcal{E} := \{\phi \in \mathrm{Aut}(\mathfrak{t});\ \phi(Z_k) \in \{\pm Z_1, \pm Z_2\} \text{ for } k = 1, 2\}.$$

We call j and j' *equivalent* if there is $A \in SU(m) \cup SU(m) \circ Q$ and $\Psi \in \mathcal{E}$ such that $j'_Z = A j_{\Psi(Z)} A^{-1}$ for all $Z \in \mathfrak{t}$.

(iii) We say that j is *generic* if no nonzero element of $\mathfrak{su}(m)$ commutes with both j_{Z_1} and j_{Z_2}.

Note that all properties above are stable under multiplication of j and j' with a fixed non-zero real number.

We will need the following proposition which is just a simplified form of [Sch01] Prop. 3.2.6(i).

Proposition 5.2.2. *For every $m \geqslant 3$ there is an open interval $I \subset \mathbb{R}$ and a continuous family $j(t)$, $t \in I$, of linear maps $\mathbb{R}^2 \to \mathfrak{su}(m)$ such that*

(i) The maps $j(t)$ are pairwise isospectral.

(ii) For $t_1, t_2 \in I$ with $t_1 \neq t_2$ the maps $j(t_1)$ and $j(t_2)$ are not equivalent.

(iii) All maps $j(t)$ are generic.

Remark. Note that the proof of (ii) in [Sch01] still holds for our slightly different definition of equivalence, since Definition 5.2.1(ii) still implies that $\mathrm{tr}((j_{Z_1}^2 + j_{Z_2}^2)^2) = \mathrm{tr}(({j'_{Z_1}}^2 + {j'_{Z_2}}^2)^2)$.

5.2.1 Isospectral Pairs

In this section we will explain how two isospectral maps $j, j' \colon \mathbb{R}^2 \to \mathfrak{su}(n-1)$ (which do not necessarily have to lie in a continuous family) induce isospectral metrics on our orbifold $\mathcal{O} = \mathcal{O}(p,q)$ from Section 5.1. More precisely, we will describe a construction process which will associate metrics $g_\lambda, g_{\lambda'}$ on $\mathcal{O}$ with j, j'.

Consider the following action of the two-torus $\widetilde{T} = S^1 \times S^1 \subset \mathbb{C}^2$ on $S^{2n+1} \subset \mathbb{C}^{n+1}$:

$$(\sigma_1, \sigma_2)(u, v_1, v_2) = (u, \sigma_1 v_1, \sigma_2 v_2) \text{ for } \sigma_1, \sigma_2 \in S^1 \subset \mathbb{C}, u \in \mathbb{C}^{n-1} \text{ and } v_1, v_2 \in \mathbb{C} \quad (5.2)$$

This action is isometric and commutes with the S^1-action above and hence induces a smooth $\widetilde{T}$-action on $\mathcal{O}$ by isometries by Corollary 2.3.14. This action is not effective but induces an effective action of

$$T := (S^1 \times S^1)/\{(\sigma, \sigma); \sigma \ p\text{-th root of unity}\}.$$

Note that the exponential map $\mathfrak{t} \ni s_1 Z_1 + s_2 Z_2 \mapsto [(e^{is_1}, e^{is_2})] \in T$ induces an isomorphism between $\mathfrak{t}/\mathcal{L}'$ and T, where $\mathcal{L}' := \mathrm{span}_{\mathbb{Z}}\{2\pi Z_1, \frac{2\pi}{p}(Z_1 + Z_2)\}$.

Moreover, set

$$\widehat{S^{2n+1}} = \{(u, v) \in \mathbb{C}^{n-1} \times \mathbb{C}^2;\ \|u\|^2 + \|v\|^2 = 1, u \neq 0, v_j \neq 0\ \forall j = 1, 2\}.$$

With $\widehat{\mathcal{O}}$ defined as in Theorem 4.1.1 (with respect to our effective T-action on $\mathcal{O} = \mathcal{O}(p,q)$) we then have $\widehat{\mathcal{O}} = P(\widehat{S^{2n+1}})$. Recall that T acts freely on the manifold $\widehat{\mathcal{O}}$ by definition.

Given a linear map $j \colon \mathbb{R}^2 \to \mathfrak{su}(n-1)$, define an $\mathbb{R}^2$-valued 1-form $\kappa = (\kappa^1, \kappa^2)$ on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by

$$\kappa^k_{(u,v)}(U, V) := \|u\|^2 \langle j_{Z_k} u, U \rangle - \langle U, iu \rangle \langle j_{Z_k} u, iu \rangle \quad (5.3)$$

for $u \in \mathbb{C}^{n-1}$, $v \in \mathbb{C}^2$, $U \in \mathbb{C}^{n-1}$ and $V \in \mathbb{C}^2$ and restricting to S^{2n+1}. Since κ is independent of V, it is T-horizontal (i.e. vanishes on $\mathfrak{t}_{(u,v)} = \{Z^*_{(u,v)};\ Z \in \mathfrak{t}\}$ for all $(u,v) \in S^{2n+1}$); in particular, $\kappa_{(u,v)}(0, iv) = \kappa_{(u,v)}(Z_1^* + Z_2^*) = 0$ for $(u,v) \in S^{2n+1}$. Moreover,

$$\kappa^k_{(u,v)}(iu, 0) = \|u\|^2 \langle j_{Z_k} u, iu\rangle - \langle iu, iu\rangle\langle j_{Z_k} u, iu\rangle = 0 \text{ for } k = 1, 2$$

(as already noted in the proof of [Sch01], 3.2.2). Hence κ is also S^1-horizontal, since the vertical space in $(u,v) \in S^{2n+1}$ under the S^1-action is given by the real span of (ipu, iqv). Moreover, κ is S^1-invariant, since S^1 acts isometrically and each $j_{Z_k} \in \mathfrak{su}(n-1)$ commutes wih scalars in $S^1 \subset \mathbb{C}$.

Note that this implies, by Theorem 2.3.9, that κ induces an $\mathbb{R}^2$-valued 1-form λ on $\mathcal{O}$ satisfying $P^*\lambda = \kappa$. In other words, if we set $[(U,V)] := P_*(U,V)$ then

$$\lambda([U,V]) = \kappa(U,V). \tag{5.4}$$

Moreover, since P^* commutes with d, we have

$$d\lambda([(U_1,V_1)], [(U_2,V_2)]) = d\kappa((U_1,V_1),(U_2,V_2)).$$

From now on when using the notation $[(U,V)] \in T_{[x]}\widehat{\mathcal{O}}$ we will always assume that $(U,V) \in T_x\widehat{S^{2n+1}}$ is S^1-horizontal.

We will need the following basic observations.

Proposition 5.2.3. *(i) $P_{|\widehat{S^{2n+1}}} \colon \widehat{S^{2n+1}} \to \widehat{\mathcal{O}}$ is $\widetilde{T} = S^1 \times S^1$-equivariant.*

(ii) For every $Z \in \mathfrak{t}$ the differential P_* maps the fundamental vector field $Z^* \in \mathcal{V}(\widehat{S^{2n+1}})$ to the fundamental vector field $\widehat{Z}$ on $\widehat{\mathcal{O}}$.

(iii) Let $j\colon \mathfrak{t} \simeq \mathbb{R}^2 \to \mathfrak{su}(n-1)$ be a linear map. Then for the $\mathfrak{t}$-valued 1-forms κ given in (5.3) and λ given in (5.4) we have:

(a) κ is $\widetilde{T}$-invariant and $\widetilde{T}$-horizontal.

(b) λ is admissible in the sense of Remark 1 in Section 4.1 with respect to the effective T-action on $\mathcal{O}$ induced by (5.2).

Proof. (i) holds by our definition of the $\widetilde{T}$-action on $\mathcal{O}$ induced by the $\widetilde{T}$-action (5.2) on S^{2n+1}. (ii) follows directly from (i).

To show (iii)(a) note that $\kappa_{(u,v)}(U,V)$ does not depend on v or V and hence is $\widetilde{T}$-invarint. We had already noted above that κ is $\widetilde{T}$-horizontal. As for (iii)(b) note that it suffices to show that λ is $\widetilde{T}$-invariant and $\widetilde{T}$-horizontal. To see $\widetilde{T}$-invariance we fix $z \in \widetilde{T}$. For $k = 1, 2$ we have $z^*\lambda^k \in \Omega^1(\mathcal{O})$; in particular, it is continuous. Since $\widehat{\mathcal{O}}$ is open and dense in $\mathcal{O}$ and z is a diffeomorphism, it suffices to show $z^*\widehat{\lambda^k} = \widehat{\lambda^k}$. But this follows from (i) and the $\widetilde{T}$-invariance of κ. Similarly, for the $\widetilde{T}$-horizontality of λ it suffices to consider $\widehat{\lambda}$. But the $\widetilde{T}$-horizontality of $\widehat{\lambda}$ follows from (ii) and the $\widetilde{T}$-horizontality of κ.

The $\widetilde{T}$-invariance and $\widetilde{T}$-horizontality of λ now imply that λ is indeed admissible with respect to the effective T-action on $\mathcal{O}$. $\square$

The following theorem is now the main result of this section. Together with the results in Section 5.3 it implies the existence of non-trivial pairs and families of isospectral metrics on $\mathcal{O} = \mathcal{O}(p,q)$.

Theorem 5.2.4. *Let $j, j' \colon \mathbb{R}^2 \to \mathfrak{su}(n-1)$ be isospectral linear maps and let λ and λ' be the corresponding admissible 1-forms on $\mathcal{O}$ given above. Then $(\mathcal{O}, g_\lambda)$ and $(\mathcal{O}, g_{\lambda'})$ are isospectral orbifolds.*

Proof. To apply Theorem 4.1.2 let $\mu \in \mathcal{L}^* \subset \mathfrak{t}^*$ and set $Z := \mu(Z_1)Z_1 + \mu(Z_2)Z_2 \in \mathfrak{t}$. Then since j and j' are isospectral, we can choose $A_Z \in SU(n-1)$ as in Definition 5.2.1(i) and set $E_\mu = (A_Z, \mathrm{Id}) \in SU(n-1) \times SU(2) \subset SU(n+1) \subset SO(2n+2)$. Then E_μ is an isometry on (S^{2n+1}, g_0) and with κ, κ' associated with j, j', respectively, according to (5.3) satisfies $\mu \circ \kappa = E_\mu^*(\mu \circ \kappa')$, as has already been shown in the proof of [Sch01] Prop. 3.2.5:

$$\begin{aligned}
(E_\mu^*(\mu \circ \kappa'))_{(u,v)}(U,V) &= (\mu \circ \kappa')_{(A_Z u, v)}(A_Z U, V) \\
&= \|A_Z u\|^2 \langle j'_Z A_Z u, A_Z U\rangle - \langle A_Z U, i A_Z u\rangle \langle j'_Z A_Z u, i A_Z u\rangle \\
&= \|u\|^2 \langle A_Z^{-1} j'_Z A_Z u, U\rangle - \langle U, iu\rangle \langle A_Z^{-1} j'_Z A_Z u, iu\rangle \\
&= \|u\|^2 \langle j_Z u, U\rangle - \langle U, iu\rangle \langle j_Z u, iu\rangle = (\mu \circ \kappa)_{(u,v)}(U,V)
\end{aligned}$$

Moreover, note that $E_\mu \in SU(n-1) \times U(2)$ is S^1-equivariant and $\widetilde{T} = S^1 \times S^1$-equivariant (since it acts as the identity on the last two components of points in $S^{2n+1} \subset \mathbb{C}^{n+1}$), hence by Theorem 2.3.12 induces a T-equivariant isometry F_μ on $(\mathcal{O}, g_0)$. This implies that for any vector X tangent to $\widehat{S^{2n+1}}$:

$$\begin{aligned}
(\mu \circ \lambda)(P_* X) &= (\mu \circ \kappa)(X) = E_\mu^*(\mu \circ \kappa')(X) = (\mu \circ \kappa')(E_{\mu *} X) \\
&= (\mu \circ P^* \lambda')(E_{\mu *} X) = (\mu \circ \lambda')(P_* E_{\mu *} X) = (\mu \circ \lambda')(F_{\mu *} P_* X) \\
&= F_\mu^*(\mu \circ \lambda')(P_* X)
\end{aligned}$$

Since $P_{|\widehat{S^{2n+1}}} \colon \widehat{S^{2n+1}} \to \widehat{\mathcal{O}}$ is a manifold submersion, this implies that F_μ satisfies condition (4.1) of Theorem 4.1.2 on $\widehat{\mathcal{O}}$. Since both sides of (4.1) are smooth, (4.1) is satisfied on $\mathcal{O}$. Since $\mu \in \mathcal{L}^*$ was arbitrary, $(\mathcal{O}, g_\lambda)$ and $(\mathcal{O}, g_{\lambda'})$ are isospectral orbifolds. $\square$

We will show in Section 5.3 that if j, j' are not equivalent and at least one of them is generic, then $(\mathcal{O}, g_\lambda)$ and $(\mathcal{O}, g_{\lambda'})$ are not isometric.

Moreover, since $\langle , \rangle$ on S^{2n+1} has constant curvature 1 and $P \colon (S^{2n+1}, \langle , \rangle) \to (\mathcal{O}, g_0)$ is a Riemannian submersion, O'Neill's curvature formula implies that after multiplying of j and j' with a sufficiently small positive real number we can assume that the metrics $g_\lambda, g_{\lambda'}$ on $\mathcal{O}^{\mathrm{reg}}$ are so close to g_0 that they have positive curvature. Therefore $(\mathcal{O}, g_\lambda)$,

$(\mathcal{O}, g_{\lambda'})$ cannot be non-trivial Riemannian product orbifolds; hence, they are not of the trivial form described above.

5.2.2 Isospectral Families

The isospectrality proof for $(\mathcal{O}, g_\lambda)$ and $(\mathcal{O}, g_{\lambda'})$ becomes considerably simpler if j, j' belong to a continuous isospectral family $j(t)$, $t \in I$. In this setting we can alternatively apply Theorem 4.1.2 directly to the sphere (with $\langle , \rangle$ replaced by a non-standard metric) to deduce that the induced metrics on the quotient are isospectral. To this end we modify $\langle , \rangle$ in such a way that the fibres of our S^1-action (5.1) become totally geodesic.

Use the standard metric $\langle , \rangle$ on S^{2n+1} to define a new metric h_0 on S^{2n+1} by setting for $(u, v) \in S^{2n+1}$, $X, Y \in T_{(u,v)}S^{2n+1}$:

$$h_0(X, Y) = (p^2\|u\|^2 + q^2\|v\|^2)^{-1}\langle X^v, Y^v\rangle + \langle X^h, Y^h\rangle,$$

where the superscripts v and h refer to the vertical and horizontal parts with respect to the given S^1-action (5.1) on $(S^{2n+1}, \langle , \rangle)$. Note that this amounts to a smooth rescaling in the vertical directions; in particular, the horizontal spaces are the same for $\langle , \rangle$ and h_0 (as are the vertical spaces, of course).

Moreover, note that the action of $\widetilde{T}$ on S^{2n+1} is still isometric with respect to h_0. In fact, the S^1-vertical distribution is $\widetilde{T}$-invariant because the $\widetilde{T}$-action and the S^1-action commuted; the rescaling function in the vertical direction is also obviously $\widetilde{T}$-invariant. Recall from Proposition 5.2.3 that if $j\colon \mathfrak{t} \simeq \mathbb{R}^2 \to \mathfrak{su}(n-1)$ is a linear map then the associated $\mathfrak{t}$-valued 1-form κ, defined as in (5.3) is $\widetilde{T}$-invariant and $\widetilde{T}$-horizontal, hence admissible with respect to the $\widetilde{T}$-action on S^{2n+1}. For such κ define $h_\kappa(X, Y) := h_0(X + \kappa(X)^*, Y + \kappa(Y)^*)$. In analogy to [Sch01] Prop. 3.2.5 (but now with the deformed metric h_0 instead of g_0) one then has:

Proposition 5.2.5. *If $j, j'\colon \mathfrak{t} \simeq \mathbb{R}^2 \to \mathfrak{su}(n-1)$ are isospectral in the sense of Definition 5.2.1(i) and κ, κ' are the corresponding $\mathfrak{t}$-valued 1-forms on S^{2n+1} given by* (5.3)*, then (S^{2n+1}, h_κ) and $(S^{2n+1}, h_{\kappa'})$ are isospectral manifolds.*

Proof. We had already recalled above how the isospectrality condition was used in [Sch01] Prop. 3.2.5 to find for each $\mu \in \mathcal{L}^*$ a map $E_\mu = (A, \mathrm{Id}) \in SU(n-1) \times SU(2) \subset SU(n+1) \subset SO(2n+2)$ satisfying $\mu \circ \kappa = E_\mu^*(\mu \circ \kappa')$. It remains to show that E_μ acts isometrically on (S^{2n+1}, h_0): E_μ commutes with the S^1-action. Therefore its differential (which is also given by E_μ) leaves the vertical spaces and hence (since it is an isometry with respect to $\langle , \rangle$) also the horizontal spaces invariant. Since the factor $(p^2\|u\|^2 + q^2\|v\|^2)^{-1}$ in the definition of h_0 is also invariant under E_μ, we deduce that h_0 is invariant under E_μ.

The proposition then follows from Theorem 4.1.2 (or from [Sch01] Thm. 1.6.). $\square$

We now write $g_\kappa(X, Y) := P^*g_\lambda(X, Y)$ and calculate, using Proposition 5.2.3 (ii), for

$x \in \widehat{S^{2n+1}}$, $X, Y \in T_x\widehat{S^{2n+1}}$:

$$\begin{aligned}
g_\kappa(X,Y) &= g_\lambda^{\mathrm{reg}}(P_*X, P_*Y)\\
&= g_0(P_*X + (\lambda^{\mathrm{reg}}(P_*X))^*_{\mathrm{reg}}(P(x)), P_*Y + (\lambda^{\mathrm{reg}}(Y))^*_{\mathrm{reg}}(P(x)))\\
&= g_0(P_*X + \kappa(X)^*_{\mathrm{reg}}(P(x)), P_*Y + \kappa(Y)^*_{\mathrm{reg}}(P(x)))\\
&= P^*g_0(X + \kappa(X)^*_x, Y + \kappa(Y)^*_x)\\
&= \langle X + \kappa(X)^*_x, Y + \kappa(Y)^*_x\rangle
\end{aligned}$$

Since $\widehat{S^{2n+1}}$ is open and dense in S^{2n+1} and both sides of the equation above are smooth, we conclude that $g_\kappa(X,Y) = \langle X + \kappa(X)^*, Y + \kappa(Y)^*\rangle$ for all vector fields X, Y on S^{2n+1}. Using this formula, we can conclude that the S^1-horizontal spaces on S^{2n+1} are the same with respect to g_κ and h_κ: For $x \in S^{2n+1}$, $X \in T_xS^{2n+1}$ and $Y \in T_x(S^1x)$ we have

$$h_\kappa(X,Y) = h_0(X + \kappa(X)^*_x, Y) \text{ and } g_\kappa(X,Y) = \langle X + \kappa(X)^*_x, Y\rangle.$$

Hence with respect to both g_κ and h_κ the horizontal space in $x \in S^{2n+1}$ is given by all $X \in T_xS^{2n+1}$ such that $X + \kappa(X)^*_x$ is horizontal with respect to $\langle , \rangle$.

Now the definition of h_0 implies that for tangent vectors X, Y which are S^1-horizontal with respect to g_κ (or, equivalently, h_κ) we have

$$h_0(X + \kappa(X)^*, Y + \kappa(Y)^*) = \langle X + \kappa(X)^*, Y + \kappa(Y)^*\rangle.$$

Therefore the induced metric $h_\kappa^{S^1}$ on our orbifold $\mathcal{O} = S^{2n+1}/S^1$ coincides with the metric g_λ from the previous section.

Proposition 5.2.6.

$$\mathrm{spec}(S^{2n+1}/S^1, h_\kappa^{S^1}) \subset \mathrm{spec}(S^{2n+1}, h_\kappa)$$

Proof. With respect to the metric h_0 all regular S^1-orbits are easily seen to have length 2π because for $(u,v) \in \widehat{S^{2n+1}}$ the length of the orbit $S^1(u,v)$ is given by

$$\begin{aligned}
&\int_0^{2\pi} \sqrt{h_0\left(\frac{d}{dt}(e^{ipt}u, e^{iqt}v), \frac{d}{dt}(e^{ipt}u, e^{iqt}v)\right)}dt\\
&\quad = \int_0^{2\pi} \sqrt{h_0\left((ipe^{ipt}u, iqe^{iqt}v), (ipe^{ipt}u, iqe^{iqt}v)\right)}dt\\
&\quad = (p^2\|u\|^2 + q^2\|v\|^2)^{-1/2} \int_0^{2\pi} \sqrt{p^2\|u\|^2 + q^2\|v\|^2}dt = 2\pi
\end{aligned}$$

Since κ is S^1-horizontal, we obtain the same length of regular S^1-orbits with respect to h_κ. Since all regular orbits have the same length, we deduce that they are totally geodesic in the manifold $(\widehat{S^{2n+1}}, h_\kappa)$. Hence the Riemannian manifold submersion $P\colon (\widehat{S^{2n+1}}, h_\kappa) \to (\widehat{S^{2n+1}}/S^1, h_\kappa^{S^1})$ has totally geodesic fibres.

This implies that given $\mu \in \mathrm{spec}(S^{2n+1}/S^1, h_\kappa^{S^1})$ and a basis $\{f_j\} \subset C^\infty(\mathcal{O})$ of the space of eigenfunctions on $(S^{2n+1}/S^1, h_\kappa^{S^1})$ to the eigenvalue μ, each restriction $f_i \circ P_{|\widehat{S^{2n+1}}}$ is an eigenfunction on $(\widehat{S^{2n+1}}, h_\kappa)$ to the eigenvalue μ ([BGM71] Prop. III.A.2.5). Since $\widehat{S^{2n+1}}$ is dense in S^{2n+1} and each $f_j \circ P$ is smooth, the $f_j \circ P$ themselves are eigenfunctions on (S^{2n+1}, h_κ) to the eigenvalue μ. Since the $f_j \circ P$ are still linearly independent, the proposition follows. $\square$

Remark. For the spectrum in the setting of Riemannian orbifold submersions with totally geodesic fibres also compare [GKP05].

Finally, we obtain the following proposition, which is actually just a special case of Theorem 5.2.4 but with an alternative proof, which does not need the orbifold version Thm. 4.1.2 of [Sch01] Thm. 1.6.

Proposition 5.2.7. *Given a continuous isospectral family of linear maps $j(t)\colon \mathfrak{t} \to \mathfrak{su}(n-1), t \in I$, the associated Riemannian metrics $h_{\kappa(t)}^{S^1} = g_{\lambda(t)}$ on $\mathcal{O} = \mathcal{O}(p,q) = S^{2n+1}/S^1$ form a continuous family of isospectral metrics on the orbifold $\mathcal{O}$.*

Proof. Denote the spectrum of $(S^{2n+1}/S^1, h_{\kappa(t)}^{S^1})$ by

$$0 = \mu_0(t) < \mu_1(t) \leqslant \mu_2(t) \leqslant \dots$$

and note that each of these functions $\mu_i\colon I \to [0,\infty)$ is continuous (as can be seen as in the compact manifold setting using Theorem 3.1.4). From Proposition 5.2.6 in connection with Proposition 5.2.5 we deduce that the image of each μ_i is discrete. Since I is connected, this implies that each μ_i is constant. In other words, the quotients $(S^{2n+1}/S^1, h_{\kappa(t)}^{S^1}) = (S^{2n+1}/S^1, g_{\kappa(t)}), t \in I$, are an isospectral family of orbifolds. $\square$

5.3 Nonisometry

In this section we will show that if j, j' are not equivalent and at least one of them is generic in the sense of Definition 5.2.1, then the corresponding metrics g_λ and $g_{\lambda'}$ on $\mathcal{O} = \mathcal{O}(p,q) = S^{2n+1}/S^1$ are not isometric. (Recall that we had fixed positive coprime integers p, q and use the action (5.1) of S^1 on S^{2n+1}.) The results on non-isometry in this section together with Proposition 5.2.2 and Theorem 5.2.4 will finally imply the main result of this thesis:

Theorem 5.3.1. *For every $n \geqslant 4$ and for all pairs (p,q) of coprime positive integers there are isospectral families of pairwise non-isometric metrics on the orbifold $\mathcal{O} = \mathcal{O}(p,q)$, a weighted projected space of dimension $2n \geqslant 8$, which is a bad orbifold for $(p,q) \neq (1,1)$.*

Some of the arguments below are based on ideas in [Rüc06]. Before we can use the criterion from Proposition 4.2.5, we need some preliminary observations. As usual, we will use the canonical metrics unless otherwise stated.

Let $\widetilde{T} = (S^1)^2$ act on $\mathbb{C}^{n-1} \setminus \{0\} \times (\mathbb{C}^*)^2$ by multiplication in the last two components and consider the following four isometric S^1-actions, where $\sigma \in S^1 \subset \mathbb{C}$, $u \in \mathbb{C}^{n-1} \setminus \{0\}$, $v \in (\mathbb{C}^*)^2$, $a, b \in \mathbb{R}_{>0}$:

- On $\mathbb{C}^{n-1}\setminus\{0\}\times(\mathbb{C}^*)^2$ set $\sigma(u,v):=(\sigma^p u,\sigma^q v)$.
- On $(\mathbb{C}^{n-1}\setminus\{0\}\times(\mathbb{C}^*)^2)/\widetilde{T}$ set $\sigma[(u,v)]:=[(\sigma^p u,\sigma^q v)]$.
- On $\mathbb{C}^{n-1}\setminus\{0\}\times(\mathbb{C}^*/S^1)^2$ set $\sigma(u,[v_1],[v_2]):=(\sigma^p u,[v_1],[v_2])$.
- On $\mathbb{C}^{n-1}\setminus\{0\}\times\mathbb{R}^2_{>0}$ set $\sigma(u,a,b):=(\sigma^p u,a,b)$.

Note that the second action above is well-defined and isometric with respect to the submersion metric by Lemma 2.3.14, because the first S^1-action and the $\widetilde{T}$-action commute.

With respect to these actions, the following isometries are S^1-equivariant:

$$(\mathbb{C}^{n-1}\setminus\{0\}\times(\mathbb{C}^*)^2)/\widetilde{T}\ni[(u,v_1,v_2)]\mapsto(u,[v_1],[v_2])\in\mathbb{C}^{n-1}\setminus\{0\}\times(\mathbb{C}^*/S^1)^2$$

and

$$\mathbb{C}^{n-1}\setminus\{0\}\times(\mathbb{C}^*/S^1)^2\ni(u,[v_1],[v_2])\mapsto(u,|v_1|,|v_2|)\in\mathbb{C}^{n-1}\setminus\{0\}\times(\mathbb{R}_{>0})^2.$$

Now recall from Subsection 5.2.1 that

$$\widehat{S^{2n+1}}=\{(u,v)\in\mathbb{C}^{n-1}\times\mathbb{C}^2;\ \|u\|^2+\|v\|^2=1,u\neq0,v_j\neq0\ \forall j=1,2\}$$

and restrict the composition of the two S^1-equivariant isometries above to the S^1-invariant submanifold $\widehat{S^{2n+1}}/\widetilde{T}$ of $(\mathbb{C}^{n-1}\setminus\{0\}\times(\mathbb{C}^*)^2)/\widetilde{T}$. Factoring out the S^1-action gives an isometry

$$\Phi\colon\ \widehat{\mathcal{O}}/\widetilde{T}\to N/S^1,$$

where

$$N:=\{(u,a,b)\in\mathbb{C}^{n-1}\setminus\{0\}\times\mathbb{R}_{>0}\times\mathbb{R}_{>0};\ \|u\|^2+a^2+b^2=1\}\subset S^{2n-1}\subset\mathbb{C}^{n-1}\times\mathbb{R}^2_{>0}.$$

Note that the S^1-actions above are not effective. However, the quotient $S^1/\{\sigma\in S^1;\ \sigma^p=1\}$ of S^1 by the p-roots of unity acts freely and it is the smooth structures induced by these free actions that we refer to. Analogously, the manifold structure on $\widehat{\mathcal{O}}/\widetilde{T}$ is induced by the free action of $T=\widetilde{T}/\{(\sigma,\sigma)\in\widetilde{T};\ \sigma^p=1\}$ on $\widehat{\mathcal{O}}$.

Let $\Pi\colon\ \widehat{\mathcal{O}}\to\widehat{\mathcal{O}}/T$ denote the quotient map and for $a,b>0$ with $a^2+b^2<1$ set

$$\begin{aligned}S_{a,b}&=(S^{2n-3}(\sqrt{1-a^2-b^2})\times\{(a,b)\})/S^1\subset N/S^1,\\ \mathcal{O}_{a,b}&=\Pi^{-1}(\Phi^{-1}(S_{a,b}))\subset\widehat{\mathcal{O}}.\end{aligned}$$

Since Π is a manifold submersion, $\mathcal{O}_{a,b}$ is a T-invariant submanifold of $\widehat{\mathcal{O}}$. By definition, under the isometry Φ the manifold $\mathcal{O}_{a,b}/T$ corresponds to

$$S_{a,b}\overset{\text{isom.}}{\simeq}(\mathbb{CP}^{n-2},(1-a^2-b^2)g_{\mathrm{FS}}),$$

where g_{FS} denotes the Fubini-Study metric on $\mathbb{CP}^{n-2}$, i.e., the submersion metric induced by the standard metric on $S^{2n-1}\subset\mathbb{C}^{n-1}$.

For $x \in \widehat{S^{2n+1}}$ let r^x denote the diffeomorphism

$$\widetilde{T} = S^1 \times S^1 \ni (\sigma_1, \sigma_2) \mapsto (\sigma_1, \sigma_2)x \in \widetilde{T}x \subset \widehat{S^{2n+1}}$$

and let $r^{[x]}$ denote the corresponding immersion

$$\widetilde{T} = S^1 \times S^1 \ni (\sigma_1, \sigma_2) \mapsto (\sigma_1, \sigma_2)[x] \in \widetilde{T}[x] \subset \widehat{S^{2n+1}}/S^1 = \widehat{\mathcal{O}}.$$

Note that $r^{[x]} = P \circ r^x$ for $P\colon \widehat{S^{2n+1}} \to \widehat{S^{2n+1}}/S^1 = \widehat{\mathcal{O}}$ the canonical projection. In the following calculations we will use our convention that on $\mathcal{O}$ the bracket $\langle , \rangle$ stands for g_0.

Proposition 5.3.2. *Let $A_j, B_j \in \mathbb{R}$, $\sigma_j \in S^1 \subset \mathbb{C}$ for $j = 1, 2$, and set*

$$A := (iA_1\sigma_1, iA_2\sigma_2), B := (iB_1\sigma_1, iB_2\sigma_2) \in T_{(\sigma_1,\sigma_2)}(S^1 \times S^1) \subset \mathbb{C}^2.$$

Moreover, let $x = (u, v) \in \widehat{S^{2n+1}}$ with $u \in \mathbb{C}^{n-1}$, $v \in \mathbb{C}^2$. Then

$$\langle r^{[x]}_* A, r^{[x]}_* B\rangle = \sum_{j=1}^{2} A_j B_j |v_j|^2 - \frac{q^2(\sum_j A_j|v_j|^2)(\sum_j B_j|v_j|^2)}{p^2\|u\|^2 + q^2\|v\|^2}.$$

Proof. First note that

$$r^x_* A = (0, iA_1\sigma_1 v_1, iA_2\sigma_2 v_2) \in T_x\widehat{S^{2n+1}}$$

(and analogously for B). The vertical space of the S^1-action on $\widehat{S^{2n+1}}$ in $(\sigma_1, \sigma_2)x$ is the $\mathbb{R}$-span of the unit vector

$$\mathbf{V} = \frac{(ipu, iq\sigma_1 v_1, iq\sigma_2 v_2)}{\sqrt{p^2\|u\|^2 + q^2\|v\|^2}} \in \mathbb{C}^{n+1}.$$

Denoting the projection onto the S^1-horizontal space with respect to $\langle , \rangle$ on S^{2n+1} by the superscript h we obtain, using $r^{[x]} = P \circ r^x$:

$$\begin{aligned}
\langle r^{[x]}_* A, r^{[x]}_* B\rangle &= \langle P_* r^x_* A, P_* r^x_* B\rangle = \langle (r^x_* A)^h, (r^x_* B)^h\rangle \\
&= \langle r^x_* A, r^x_* B\rangle - \langle r^x_* A, \mathbf{V}\rangle\langle r^x_* B, \mathbf{V}\rangle \\
&= \sum_j A_j B_j |v_j|^2 - \frac{q^2(\sum_j A_j|v_j|^2)(\sum_j B_j|v_j|^2)}{p^2\|u\|^2 + q^2\|v\|^2}.
\end{aligned}$$

□

Recall that $Z_1 = (i, 0), Z_2 = (0, i)$ denote the standard basis of $\mathfrak{t} = T_{(1,1)}(S^1 \times S^1) \subset \mathbb{C}^2$ and that for $Z \in \mathfrak{t}$ the symbol $\widehat{Z}$ denotes the fundamental manifold vector field associated with Z with respect to the action of T (or, equivalently, $\widetilde{T}$) on $\widehat{\mathcal{O}}$. Moreover, note that

Proposition 5.2.3(i) implies

$$\widehat{Z_k} \circ P = P_* \circ Z_k^* = P_* r^{\cdot}_{*(1,1)} Z_k = r^{[\cdot]}_{*\,(1,1)} Z_k.$$

Corollary 5.3.3. *For $j, k \in \{1,2\}$ and $x = (u,v) \in \widehat{S^{2n+1}}$ we have*

$$\langle \widehat{Z}_{j[x]}, \widehat{Z}_{k[x]} \rangle = \delta_{jk} |v_j|^2 - \frac{q^2 |v_j|^2 |v_k|^2}{p^2 \|u\|^2 + q^2 \|v\|^2}.$$

Proof. Apply Proposition 5.3.2 to $\widehat{Z}_{1[x]} = r^{[x]}_{*\,(1,1)} Z_1 = r^{[x]}_{*\,(1,1)} (i,0)$ and $\widehat{Z}_{2[x]} = r^{[x]}_{*\,(1,1)} Z_2 = r^{[x]}_{*\,(1,1)} (0,i)$ in $\sigma = (1,1)$. □

Corollary 5.3.4. *For $[x] \in \mathcal{O}_{a,a}$ we have*

(i)

$$\langle \widehat{Z}_{j[x]}, \widehat{Z}_{k[x]} \rangle = \delta_{jk} a^2 - \frac{q^2 a^4}{p^2(1-2a^2) + 2q^2 a^2}$$

(ii)

$$\angle(\widehat{Z}_{1[x]}, \widehat{Z}_{2[x]}) = \arccos \frac{-q^2 a^2}{p^2(1-2a^2) + q^2 a^2}.$$

Proof. (i) follows directly from Corollary 5.3.3. (ii) follows from (i).

$$\begin{aligned}
\|\widehat{Z}_{1[x]}\|^2 = \|\widehat{Z}_{2[x]}\|^2 &= a^2 - \frac{q^2 a^4}{p^2(1-2a^2) + 2q^2 a^2} = \frac{p^2(1-2a^2)a^2 + 2q^2 a^4 - q^2 a^4}{p^2(1-2a^2) + 2q^2 a^2} \\
&= \frac{a^2(p^2(1-2a^2) + q^2 a^2)}{p^2(1-2a^2) + 2q^2 a^2}, \\
\langle \widehat{Z}_{1[x]}, \widehat{Z}_{2[x]} \rangle &= -\frac{q^2 a^4}{p^2(1-2a^2) + 2q^2 a^2}
\end{aligned}$$

Hence

$$\cos \angle(\widehat{Z}_{1[x]}, \widehat{Z}_{2[x]}) = \frac{\langle \widehat{Z}_{1[x]}, \widehat{Z}_{2[x]} \rangle}{\|\widehat{Z}_{1[x]}\| \|\widehat{Z}_{2[x]}\|} = \frac{-q^2 a^2}{p^2(1-2a^2) + q^2 a^2}.$$

□

Moreover, we will soon need the following observation.

Lemma 5.3.5. *Given $a, b > 0$ with $a^2 + b^2 < 1$ and $[x] \in \mathcal{O}_{a,b}$, the map*

$$f^{[x]} \colon T \ni z \mapsto z[x] \in T[x] \subset \mathcal{O}_{a,b} \subset \widehat{\mathcal{O}}$$

is an embedding and the pull-back by $f^{[x]}$ of the metric $g_0 = \langle , \rangle$ to T is left-invariant and associated with the inner product

$$(Y_1, Y_2) \mapsto \left\langle \widehat{Y}_{1[x]}, \widehat{Y}_{2[x]} \right\rangle$$

.

Proof. This follows, since T is abelian and acts by isometries. $\square$

We will now apply the methods from Section 4.2 to our T-action on $\mathcal{O} = \mathcal{O}(p,q)$ with metric $g_0 = \langle , \rangle$ to show non-isometry of our examples under certain conditions. Recall from Notations and Remarks 4.2.1(ii) that $\mathrm{Aut}^T_{g_0}(\mathcal{O})$ is the group of all T-preserving diffeomorphisms of $\mathcal{O}$ which preserve the g_0-norm of vectors tangent to the T-orbits in $\widehat{\mathcal{O}}$ and induce an isometry of $(\widehat{\mathcal{O}}/T, g_0^T)$.

We first use the formulas above to show the following lemma, from which we will need only the case $a = b$ in the proof of Proposition 5.3.7.

Lemma 5.3.6. *Let $a, b > 0$ with $a^2 + b^2 < 1$ and $F \in \mathrm{Aut}^T_{g_0}(\mathcal{O})$. Then $F(\mathcal{O}_{a,b} \cup \mathcal{O}_{b,a}) = \mathcal{O}_{a,b} \cup \mathcal{O}_{b,a}$.*

Proof. For $c \in (0,1)$ set

$$\mathcal{O}_c = \bigcup_{\substack{r^2+s^2=1-c^2\\ r,s>0}} \mathcal{O}_{r,s} \subset \widehat{\mathcal{O}}.$$

We proceed in two steps.

First step: We will first show that F preserves every $\mathcal{O}_c$. To this end set for each $c \in (0,1)$:

$$N_c := S^{2n-3}(c) \times \{(r,s) \in (\mathbb{R}_{>0})^2;\ r^2 + s^2 = 1 - c^2\} \subset S^{2n-1} \subset \mathbb{C}^{n-1} \times \mathbb{R}^2.$$

and observe that

$$\Phi(\mathcal{O}_c/T) = N_c/S^1$$

and $N = \bigcup_{c\in(0,1)} N_c$.

Now fix $c \in (0,1)$. Note that $\mathcal{O}_c$ is T-invariant and hence F leaves $\mathcal{O}_c$ invariant if and only if the induced isometry $\bar{F} \in \overline{Aut}^T_{g_0}(\mathcal{O})$ of $\widehat{\mathcal{O}}/T$ leaves $\mathcal{O}_c/T$ invariant. The isometry $\Phi\colon \widehat{\mathcal{O}}/T \to N/S^1$ has a unique continuous extension

$$\widetilde{\Phi}\colon \mathcal{O}/T = \widetilde{\widehat{\mathcal{O}}}/T = \widetilde{\widehat{\mathcal{O}}/T} \to \widetilde{N/S^1} = \widetilde{N}/S^1 = \overline{N}/S^1,$$

where the tildes denote the completions of the respective metric spaces. This extension is again given by $(S^{2n+1}/S^1)/T \ni [(u, v_1, v_2)] \mapsto [(u, |v_1|, |v_2|)]$. Write $\widetilde{\Pi}\colon \mathcal{O} \to \mathcal{O}/T$ for the canonical projection and note that $\widetilde{\Pi}$ is the unique continuous extension of $\Pi\colon \widehat{\mathcal{O}} \to \widehat{\mathcal{O}}/T$. Moreover, note that

$$\overline{N} = \{(u,r,s) \in \mathbb{C}^{n-1} \times (\mathbb{R}_{\geqslant 0})^2; \|u\|^2 + r^2 + s^2 = 1\} \subset S^{2n-1} \subset \mathbb{C}^{n-1} \times \mathbb{R}^2.$$

Extend $\bar{F} \in \mathrm{Isom}(\widehat{\mathcal{O}}/T)$ uniquely to a metric space isometry $\widetilde{F}$ of $\mathcal{O}/T$ and note that $\widetilde{F} \circ \widetilde{\Pi} = \widetilde{\Pi} \circ F$ by continuity. We will show that $\widetilde{\Phi} \circ \widetilde{F} \circ \widetilde{\Phi}^{-1} \in \mathrm{Isom}(\overline{N}/S^1)$ preserves N_c/S^1: Set

$$N_1 := S^{2n-3} \times \{(0,0)\} \subset \overline{N}.$$

Then

(i) N_1/S^1 is invariant by the isometry $\widetilde{\Phi} \circ \widetilde{F} \circ \widetilde{\Phi}^{-1}$: Since F maps T-orbits in $\mathcal{O}$ to T-orbits, it preserves

$$\widetilde{\Pi}^{-1}(\widetilde{\Phi}^{-1}(N_1/S^1)) = \{(u,0,0) \in \mathbb{C}^{n-1} \times \mathbb{C}^2;\ \|u\| = 1\}/S^1 \subset \mathcal{O}$$

because this is the only $(2n-4)$-dimensional component of the union of all T-orbits in $\mathcal{O}$ with only one element. But this implies that $\widetilde{F}$ preserves $\widetilde{\Phi}^{-1}(N_1/S^1)$.

(ii) Each N_c/S^1 ($c \in (0,1)$) is precisely the set of all points in $\widetilde{N}/S^1$ which have distance $\arccos c$ (in radians) from N_1/S^1: For $x = (u,r,s) \in N_c$ and $x' = (u',0,0) \in N_1$ the distance between x and x' in $\widetilde{N} = \overline{N} \subset S^{2n-1}$ is given by the angle $\angle(x,x')$. Since

$$\langle x, x'\rangle = \langle u, u'\rangle \leqslant \|u\|\|u'\| = \|u\| = c$$

and $\langle x, (\frac{u}{c},0,0)\rangle = c$, we have $\operatorname{dist}(x,N_1) = \arccos c$ for all $x \in N_c$ and hence for all $[x] \in N_c/S^1$:

$$\begin{aligned}
\operatorname{dist}([x], N_1/S^1) &= \min_{[y]\in N_1/S^1} d([x],[y]) = \min_{[y]\in N_1/S^1} \min_{\sigma\in S^1} d(\sigma x, y) \\
&= \min_{y\in N_1} \min_{\sigma\in S^1} d(\sigma x, y) = \min_{\sigma\in S^1} \min_{y\in N_1} d(\sigma x, y) \\
&= \min_{\sigma\in S^1} \operatorname{dist}(\sigma x, N_1) = \min_{\sigma\in S^1} \arccos c \\
&= \arccos c.
\end{aligned}$$

(i) and (ii) together imply that the isometry $\widetilde{\Phi} \circ \widetilde{F} \circ \widetilde{\Phi}^{-1}$ leaves N_c/S^1 invariant, hence $\bar{F} = \widetilde{F}_{|\widehat{\mathcal{O}}/T}$ preserves $\mathcal{O}_c/T$ and therefore $\mathcal{O}_c$ is invariant under F.

Second step: Now let $a, b > 0$ with $a^2 + b^2 < 1$ and $F \in \operatorname{Aut}^T_{g_0}(\mathcal{O})$ be as in the lemma.

Fix $[x] = [(u,v)] \in \mathcal{O}_{a,b}$. Recall from Lemma 5.3.5 that the pull-back by $f^{[x]}$ of the metric g_0 to T is left-invariant and associated with the inner product

$$(Y_1, Y_2) \mapsto g_0\left(\widehat{Y_1}_{[x]}, \widehat{Y_2}_{[x]}\right) = \left\langle \widehat{Y_1}_{[x]}, \widehat{Y_2}_{[x]}\right\rangle$$

on $\mathfrak{t}$.

Moreover, note that the area of T with respect to its standard bi-invariant metric (with $\{Z_1, Z_2\}$ an orthonormal basis of $\mathfrak{t}$) is $4\pi^2/p$, since $T \simeq \mathfrak{t}/\mathcal{L}'$ with

$$\mathcal{L}' = \operatorname{span}_{\mathbb{Z}}\left\{2\pi Z_1, \frac{2\pi}{p}(Z_1 + Z_2)\right\}.$$

Hence, the area of $T[x]$ is given by

$$A(T[x]) = \frac{4\pi^2}{p}\sqrt{\det(\langle \widehat{Z_j}_{[x]}, \widehat{Z_k}_{[x]}\rangle)_{j,k=1,2}}.$$

Set $c = \|u\| = \sqrt{1-a^2-b^2}$ so that $\mathcal{O}_{a,b} \cup \mathcal{O}_{b,a} \subset \mathcal{O}_c$. Corollary 5.3.3 gives

$$\begin{aligned}\frac{p^2}{16\pi^4}A(T[x])^2 &= \left(a^2 - \frac{q^2a^4}{p^2c^2+q^2(1-c^2)}\right)\left(b^2 - \frac{q^2b^4}{p^2c^2+q^2(1-c^2)}\right)\\ &\quad - \left(\frac{q^2a^2b^2}{p^2c^2+q^2(1-c^2)}\right)^2\\ &= a^2b^2 - \frac{q^2a^2b^2(1-c^2)}{p^2c^2+q^2(1-c^2)} = a^2b^2\left(1 - \frac{q^2(1-c^2)}{p^2c^2+q^2(1-c^2)}\right)\end{aligned}$$

Note that since F preserves the length of vectors tangent to T-orbits by definition, we have $A(T[x]) = A(F(T[x])) = A(TF([x]))$. Moreover, we had seen in the first step that $\mathcal{O}_c$ is invariant under F. These two observations and the equation above then imply that for $F([x]) =: [(u', v_1', v_2')]$ and $a' := |v_1'|, b' := |v_2'|$, we have ${a'}^2 + {b'}^2 = a^2 + b^2$ and ${a'}^2{b'}^2 = a^2b^2$. This implies $a + b = a' + b'$ and $(a-b)^2 = (a'-b')^2$. These two equations in turn show that $a = a' \ \wedge \ b = b'$ or $a = b' \ \wedge \ b = a'$. In other words, F preserves $\mathcal{O}_{a,b} \cup \mathcal{O}_{b,a}$. Since F^{-1} also lies in $\mathrm{Aut}^T_{g_0}(\mathcal{O})$, the lemma follows. □

Now recall that we had set $\mathcal{D} := \{\Psi_F;\ F \in \mathrm{Aut}^T_{g_0}(\mathcal{O})\} \subset \mathrm{Aut}(\mathfrak{t})$ in Notations and Remarks 4.2.1 (iii) and $\mathcal{E} := \{\phi \in \mathrm{Aut}(\mathfrak{t});\ \phi(Z_k) \in \{\pm Z_1, \pm Z_2\}\ \forall k = 1,2\}$ in Definition 5.2.1(ii). We are now in a position to show that in our special case we have the following inclusion.

Proposition 5.3.7.

$$\mathcal{D} \subset \mathcal{E}$$

Proof. Let $F \in \mathrm{Aut}^T_{g_0}(\mathcal{O})$. We have to show that $\Psi_F(Z_k) \in \{\pm Z_1, \pm Z_2\}$ for $k = 1,2$. By 4.2.1 we know $F_*(\widehat{Z_k}) = \widehat{\Psi_F(Z_k)}$ on $\widehat{\mathcal{O}}$. The map

$$\mathfrak{t} \ni Z \mapsto \widehat{Z}_{[x]} \in T_{[x]}\widehat{\mathcal{O}}$$

is injective for any $[x] \in \widehat{\mathcal{O}}$, because $T \ni z \mapsto z[x] \in T[x]$ is a diffeomorphism. So it suffices to show that $F_{*[x]}(\widehat{Z}_{k[x]}) \in \{\pm\widehat{Z}_{1F([x])}, \pm\widehat{Z}_{2F([x])}\}$ for $k = 1,2$ in a single point $[x] \in \widehat{\mathcal{O}}$.

Since the expression in Corollary 5.3.4 (ii) is continuous and non-constant in a, we can choose $a \in (0, \frac{1}{\sqrt{2}})$ such that for all $[x] \in \mathcal{O}_{a,a}$:

$$\cos\angle(\widehat{Z}_{1[x]}, \widehat{Z}_{2[x]}) \in \mathbb{R} \setminus \mathbb{Q}.$$

Now let $[x] \in \mathcal{O}_{a,a}$ be arbitrary. Temporarily write $\langle Y_1, Y_2\rangle := \langle \widehat{Y}_{1[x]}, \widehat{Y}_{2[x]}\rangle$ and $\|Y\| := \sqrt{\langle Y, Y\rangle}$ for $Y_1, Y_2, Y \in \mathfrak{t}$. Note that $\|Z_1\| = \|Z_2\|$. Since for $k, l \in \mathbb{Z}$ we have

$$\frac{\|kZ_1 + lZ_2\|^2}{\|Z_1\|^2} = k^2 + l^2 + 2kl\frac{\langle Z_1, Z_2\rangle}{\|Z_1\|^2},$$

we deduce (by our choice of a) that

$$\forall k, l \in \mathbb{Z}\colon \left(\frac{\|kZ_1 + lZ_2\|^2}{\|Z_1\|^2} \in \mathbb{Q} \Rightarrow kl = 0 \right).$$

Hence if $Y \in \mathcal{L}'$ with $\|Y\| = \|Z_1\| = \|Z_2\|$, then (since $pY \in \mathcal{L}$ and hence $pY = kZ_1 + lZ_2$ for some $k, l \in \mathbb{Z}$) we have $pY \in \{\pm pZ_1, \pm pZ_2\}$ and hence $Y \in \{\pm Z_1, \pm Z_2\}$.

This implies that the images of the two flow lines generated by $\widehat{Z_1}$ and $\widehat{Z_2}$ through $[x]$ give precisely the geodesic loops in $T[x] \subset \mathcal{O}_{a,a}$ through $[x]$ of length $2\pi\|Z_1\|$; recall from Lemma 5.3.5 that $T \ni z \mapsto z[x] \in T[x]$ is an isometry with respect to some left-invariant mectric on T, hence such flow lines are indeed geodesics.

Since F preserves $\mathcal{O}_{a,a}$ by Lemma 5.3.6, we have $F(T[x]) \subset \mathcal{O}_{a,a}$ and the geodesic loops in $TF([x]) = F(T[x])$ through $F([x])$ of length $2\pi\|Z_1\|$ are given precisely by the flow lines of $\widehat{Z_1}$ and $\widehat{Z_2}$ through $F([x])$. On the other hand, since $F\colon T[x] \to F(T[x])$ is an isometry, the images of the flow lines of $F_*\widehat{Z_1}$ and $F_*\widehat{Z_2}$ through $F([x])$ in $F(T[x])$ also have length $2\pi\|Z_1\|$. Together this implies

$$F_{*[x]}(\widehat{Z_j}_{[x]}) \in \{\pm \widehat{Z_1}_{F([x])}, \pm \widehat{Z_2}_{F([x])}\}$$

for $j = 1, 2$. As noted above, this proves our statement. $\square$

Now recall the following criterion for nonisometry (Proposition 4.2.5):

Proposition 5.3.8. *Let λ, λ' be admissible 1-forms on the orbifold $\mathcal{O}$ such that*

(N) $\Omega_\lambda \notin \mathcal{D} \circ \overline{Aut}^T_{g_0}(\mathcal{O})^* \Omega_{\lambda'}$.

(G) No nontrivial 1-parameter group in $\overline{Aut}^T_{g_0}(\mathcal{O})$ preserves $\Omega_{\lambda'}$.

Then $(\mathcal{O}, g_\lambda)$ and $(\mathcal{O}, g_{\lambda'})$ are not isometric.

The proposition above and the following proposition will imply that if isospectral maps j and j' satisfy the conditions from the proposition below, the corresponding isospectral orbifolds $(\mathcal{O}, g_\lambda)$, $(\mathcal{O}, g_{\lambda'})$ with $\mathcal{O} = \mathcal{O}(p, q)$ are non-isometric. In its proof we basically follow [Sch01] Prop. 4.3.

Proposition 5.3.9. *Let $j, j'\colon \mathbb{R}^2 \to su(n-1)$ be two linear maps and let λ, λ' be the admissible $\mathfrak{t}$-valued 1-forms on $\mathcal{O} = \mathcal{O}(p, q)$ associated with j and j'.*

(i) If j and j' are not equivalent in the sense of Definition 5.2.1 (ii), then Ω_λ and $\Omega_{\lambda'}$ satisfy condition (N).

(ii) If j is generic in the sense of Definition 5.2.1 (iii), then Ω_λ has property (G).

Proof. Choose an arbitrary $a \in (0, 1/\sqrt{2})$ and set $L := \mathcal{O}_{a,a} \subset \widehat{\mathcal{O}}$. Moreover, for a $\mathfrak{t}$-valued k-form η on an manifold we define real-valued k-forms on M by $\eta =: \eta^1 Z_1 + \eta^2 Z_2$. We write Ω_0^L for the $\mathfrak{t}$-valued 2-form on L/T induced by the curvature form Ω_0 on $(\widehat{\mathcal{O}}/T, g_0^T)$.

First step: Calculation of Ω_0^L: In this step we will show that on $L/T \overset{\text{isom}}{\simeq} (\mathbb{CP}^{n-2}, (1-2a^2)g_{\text{FS}})$ each form $(\Omega_0^L)^j$, $j = 1, 2$, is a nonvanishing multiple of the standard Kähler form.

Recall from 4.2.1 (iv) that $\omega_0 \colon T\widehat{\mathcal{O}} \to \mathfrak{t}$ denotes the connection form on the principal T-bundle $\widehat{\mathcal{O}}$ associated with g_0. We will first show that with $P \colon \widehat{S^{2n+1}} \to \widehat{\mathcal{O}}$ the canonical projection we have for $(u, v) \in \widehat{S^{2n+1}}$, $X = (U, V) \in T_{(u,v)}\widehat{S^{2n+1}}$, $j = 1, 2$:

$$(P^*{\omega_0}^j)_{(u,v)}(X) = -\frac{q}{p}\frac{\langle U, iu\rangle}{\|u\|^2} + \frac{\langle V_j, iv_j\rangle}{|v_j|^2} \tag{5.5}$$

Let ${\eta_0}^j_{(u,v)}(X)$ denote the term on the right hand side. Then the 1-form η_0^j on $\widehat{S^{2n+1}}$ is easily seen to be S^1-invariant. It is also S^1-horizontal, since the vertical space in $(u, v) \in \widehat{S^{2n+1}}$ is given by the $\mathbb{R}$-span of (ipu, iqv_1, iqv_2) and

$$\eta_0^j(ipu, iqv_1, iqv_2) = -\frac{q}{p}\frac{\langle ipu, iu\rangle}{\|u\|^2} + \frac{\langle iqv_j, iv_j\rangle}{|v_j|^2} = -q + q = 0.$$

Hence η_0^j in the pull-back of a 1-form on $\widehat{\mathcal{O}}$. Moreover, since ${Z_1}^*_{(u,v)} = (0, iv_1, 0)$, ${Z_2}^*_{(u,v)} = (0, 0, iv_2)$ on S^{2n+1}, we observe that $\eta_0^j(Z_k) = \delta_{jk}$. Since P is a Riemannian submersion, it remains to show that for $(u, v) \in \widehat{S^{2n+1}}$ the form η_0^j vanishes on the space W of all (U, V) in $T_{(u,v)}\widehat{S^{2n+1}}$ which are perpendicular to ${Z_1}^*_{(u,v)}$ and ${Z_2}^*_{(u,v)}$ and which are S^1-horizontal. But the first two conditions imply $V = 0$ and the last condition finally implies that $W = \{(U, 0, 0) \in T_{(u,v)}\widehat{S^{2n+1}}; U \perp iu\}$. Since η_0^j obviously vanishes on W, we conclude that $P^*\omega_0^j = \eta_0^j$, i.e., we have established (5.5).

Now write ω_0^L for the $\mathfrak{t}$-valued 1-form on L induced by ω_0. (5.5) implies that for $(u, v) \in L$ and $X \in T_{(u,v)}P^{-1}(L)$:

$$(P^*\omega_0^L)^j(X) = -\frac{q}{p(1-2a^2)}\langle U, iu\rangle + \frac{\langle V_j, iv_j\rangle}{a^2}.$$

Now note that $P^{-1}(L) = S^{2n-3}(\sqrt{1-2a^2}) \times (S^1(a))^2 \subset S^{2n+1} \subset \mathbb{C}^{n+1}$ and hence if $X = (U, V) \in T_{(u,v)}P^{-1}(L)$, then V_j is a real multiple of iv_j for $j = 1, 2$. Using this, the equation above implies that for $X = (U, V), \widetilde{X} = (\widetilde{U}, \widetilde{V})$ tangent to $P^{-1}(L)$ in (u, v) and $j = 1, 2$:

$$\begin{aligned}
&(P^*d\omega_0^L)^j(X, \widetilde{X})\\
&\qquad = -\frac{q}{p(1-2a^2)}\langle \widetilde{U}, iU\rangle + \frac{\langle \widetilde{V}_j, iV_j\rangle}{p(1-2a^2)} + \frac{q}{p(1-2a^2)}\langle U, i\widetilde{U}\rangle - \frac{\langle V_j, i\widetilde{V}_j\rangle}{p(1-2a^2)}\\
&\qquad = -2\frac{q}{p(1-2a^2)}\langle iU, \widetilde{U}\rangle
\end{aligned}$$

Therefore, on $L/T \overset{\text{isom}}{\simeq} (\mathbb{CP}^{n-2}, (1-2a^2)g_{\text{FS}})$ the form $(\Omega_0^L)^j$ is a nonvanishing multiple of the standard Kähler form.

Second step: Proof of (i): Suppose that condition (N) is not satisfied. Then there is $\Psi \in \mathcal{D}$ and $F \in \mathrm{Aut}^T_{g_0}(\mathcal{O})$ such that $\Omega_\lambda = \Psi \circ \overline{F}^* \Omega_{\lambda'}$. Since $\overline{F}$ preserves L/T (by Lemma 5.3.6), this implies $\Omega^L_\lambda = \Psi \circ \overline{F}^* \Omega^L_{\lambda'}$. Now $\Omega_\lambda = \Omega_0 + d\overline{\lambda}$ and $\Omega_{\lambda'} = \Omega_0 + d\overline{\lambda'}$ (4.2.1(vi)) imply (with $\overline{\lambda}^L$ denoting the $\mathfrak{t}$-valued 1-form on L/T induced by $\overline{\lambda}$, and analogously for $\overline{\lambda'}$):

$$\Omega^L_0 + d\overline{\lambda}^L = \Omega^L_\lambda = \Psi \circ \overline{F}^* \Omega^L_{\lambda'} = \Psi \circ \overline{F}^* (\Omega^L_0 + d\overline{\lambda'}^L). \tag{5.6}$$

In particular, $\Omega^L_0 - \Psi \circ \overline{F}^* \Omega^L_0$ is exact. Moreover, note that Proposition 5.3.7 implies $\Psi \in \mathcal{E}$. From $(\Omega^L_0)^1 = (\Omega^L_0)^2$ we then conclude $\Omega^L_0 - \Psi \circ \overline{F}^* \Omega^L_0 \in \{0, 2\Omega^L_0\}$. However, by the first step above, $2\Omega^L_0$ cannot be exact, and therefore $\Omega^L_0 - \Psi \circ \overline{F}^* \Omega^L_0 = 0$. (5.6) then implies

$$d\overline{\lambda}^L = \Psi \circ \overline{F}^* d\overline{\lambda'}^L. \tag{5.7}$$

Let $Q\colon \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ denote complex conjugation and choose $A \in SU(n-1) \cup SU(n-1) \circ Q$ such that A induces (via the Hopf fibration $\mathbb{C}^{n-1} \supset S^{2n-3} \to \mathbb{CP}^{n-2}$) the isometry on $L/T \simeq (\mathbb{CP}^{n-2}, (1-2a^2)g_{\mathrm{FS}})$ corresponding to $\overline{F}_{|L/T}$, i.e., such that $P \circ (A, I_2)_{|P^{-1}(L)} = F \circ P_{|P^{-1}(L)}$. Then, with κ^L denoting the restriction of κ to $P^{-1}(L)$ (and analogously for κ'), (5.7) implies, by $\kappa = P^*\lambda$, $\kappa' = P^*\lambda'$:

$$d\kappa^L = \Psi \circ (A, I_2)^* d{\kappa'}^L. \tag{5.8}$$

For $k \in \{1, 2\}$ set $j_k := j_{Z_k}$. For $(u, v) \in P^{-1}(L)$ we have

$$\kappa^k_{(u,v)}(U, V) = (1 - 2a^2)\langle j_k u, U\rangle - \langle U, iu\rangle\langle j_k u, iu\rangle$$

Now let $(U_1, V_1), (U_2, V_2) \in T_{(u,v)}\widehat{S^{2n+1}}$. Then we get by elementary differentiation and skew-symmetry:

$$\begin{aligned} d\kappa^k_{(u,v)}((U_1, V_1), (U_2, V_2)) = {} & 2(1 - 2a^2)\langle j_k U_1, U_2\rangle - 2\langle iU_1, U_2\rangle\langle j_k u, iu\rangle \\ & - 2\langle U_2, iu\rangle\langle j_k U_1, iu\rangle + 2\langle U_1, iu\rangle\langle j_k U_2, iu\rangle \end{aligned} \tag{5.9}$$

Denote by U^h the orthogonal projection of $U \in T_u\mathbb{C}^{n-1}$ to $(iu)^\perp$. Then we have $U_1 = U^h_1 + \frac{\langle U_1, iu\rangle}{1-2a^2} iu$ (and analogously for U_2) and hence

$$\begin{aligned} 2(1 - 2a^2)\langle j_k U_1, U_2\rangle = {} & 2(1 - 2a^2)\langle j_k U^h_1, U^h_2\rangle + 2\langle j_k U_1, iu\rangle\langle U_2, iu\rangle \\ & + 2\langle U_1, iu\rangle \underbrace{\langle j_k(iu), U_2\rangle}_{= -\langle j_k U_2, iu\rangle}, \end{aligned} \tag{5.10}$$

where in the last two summands we have used that $j_k(iu) \perp iu$. (5.9) and (5.10) now imply

$$d\kappa^k_{(u,v)}((U_1, V_1), (U_2, V_2)) = 2(1 - 2a^2)\langle j_k U^h_1, U^h_2\rangle - 2\langle j_k u, iu\rangle\langle iU_1, U_2\rangle. \tag{5.11}$$

Now choose $\varepsilon_k \in \{\pm 1\}$ and $l \in \{1,2\}$ such that $\Psi(Z_k) = \varepsilon_k Z_l$. Plugging (5.11) and the analogous formula for κ', j' into (5.8) we obtain, since A either commutes or anticommutes with i:

$$\begin{aligned}
2(1-2a^2)\langle j_l U_1^h, U_2^h\rangle - 2\langle j_l u, iu\rangle\langle iU_1, U_2\rangle &= d\kappa^l_{(u,v)}((U_1,V_1),(U_2,V_2)) \\
&= (\Psi \circ (A, I_2)^* d\kappa')^l((U_1,V_1),(U_2,V_2)) \\
&= (\Psi \circ d\kappa'_{(Au,v)})^l((AU_1,V_1),(AU_2,V_2)) \\
&= \varepsilon_k d{\kappa'}^k_{(Au,v)}((AU_1,V_1),(AU_2,V_2)) \\
&= 2\varepsilon_k((1-2a^2)\langle j'_k AU_1^h, AU_2^h\rangle - \langle j'_k Au, iAu\rangle\langle iAU_1, AU_2\rangle) \\
&= 2\varepsilon_k((1-2a^2)\langle A^{-1}j'_k AU_1^h, U_2^h\rangle - \langle A^{-1}j'_k Au, iu\rangle\langle iU_1, U_2\rangle).
\end{aligned}$$

Setting $\tau_k := \varepsilon_k A^{-1} j'_k A - j_l \in \mathfrak{su}(n-1)$ gives

$$0 = (1-2a^2)\langle \tau_k U_1^h, U_2^h\rangle - \langle \tau_k u, iu\rangle\langle iU_1, U_2\rangle.$$

Plugging in $U_2 = iU_1$, we observe that for $U_1 \in \operatorname{span}\{u, iu\}^\perp \setminus \{0\} \subset \mathbb{C}^{n-1}$ we have

$$\frac{\langle \tau_k U_1, iU_1\rangle}{\|U_1\|^2} = \frac{\langle \tau_k u, iu\rangle}{1-2a^2} = \frac{\langle \tau_k u, iu\rangle}{\|u\|^2}$$

Hence the map $\phi\colon\ \mathbb{C}^{n-1}\setminus\{0\} \ni U \mapsto \frac{\langle i\tau_k U, U\rangle}{\|U\|^2} \in \mathbb{R}$ is constant, say C, on $\operatorname{span}\{u,iu\}^\perp\setminus\{0\}$ and $\phi(u) = \phi(iu) = C$. Since $i\tau_k$ is hermitian, it follows elementarily that $\phi = C$ on all of $\mathbb{C}^{n-1}\setminus\{0\}$ (just decompose an arbitrary vector into its components with respect to $u, iu, \{u, iu\}^\perp$). Hence, all eigenvalues of $i\tau_k$ equal C, and thus $i\tau_k = CI_{n-1}$. Since τ_k has trace zero, we conclude $\tau_k = 0$. This finally implies $j_{\Psi(Z_k)} = A^{-1} j'_{Z_k} A$ for $k = 1,2$ and therefore $j_{\Psi(Z)} = A^{-1} j'_Z A$. In other words, j and j' are equivalent.

Third step: Proof of (ii): Assume that Ω_λ does not satisfy property (G). Then there is a non-trivial one-parameter family $\bar{F}_t \in \overline{\mathrm{Aut}}^T_{g_0}(\mathcal{O})$ such that $\bar{F}_t^*\Omega_\lambda = \Omega_\lambda$ for all t. The same argument as above (with $\Psi = \mathrm{Id}$ and $\kappa = \kappa'$) gives a one-parameter family $A_t \in SU(n-1) \cup SU(n-1) \circ Q$ such that (A_t, I_2) preserves $d\kappa^L$. Note that $A_0 = \mathrm{Id}$ implies $A_t \in SU(n-1)$. As in the proof of (i) the relation $(A_t, I_2)^* d\kappa^L = d\kappa^L$ implies $j_Z = A_t j_Z A_t^{-1}$. Taking the derivative with respect to t in 0 gives $0 = [\dot{A}_0, j_Z]$ for all $Z \in \mathfrak{t}$ in contradiction to the genericity assumption. □

5.4 Isospectral Quotients of Weighted Projective Spaces

In this section we will apply the construction from [Sut06] to give isospectral metrics on quotients of the form $\mathcal{O}(p,q)/G$ with $\mathcal{O}(p,q)$ from the preceding sections and G now a finite subgroup of the given 2-torus T (which was introduced in 5.2.1). Note that these Riemannian orbifolds are still bad.

We first phrase a special case of Sutton's results on equivariant isospectrality ([Sut06]) for orbifolds. Note that the notation from Theorem 3.2.4 easily carries over to Riemannian orbifolds and recall that a quotient of a Riemannian orbifold by a finite subgroup of the isometry group carries a canonical Riemannian orbifold structure (Thm. 2.2.4, Cor. 2.3.18). In this context the following theorem holds.

Theorem 5.4.1. *Let G be a finite group acting effectively and isometrically on two compact Riemannian orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the latter are equivariantly isospectral with respect to G, i.e., such that there is a unitary isomorphism $U\colon L^2(\mathcal{O}_1) \to L^2(\mathcal{O}_2)$ between the G-representations τ_1^G and τ_2^G (given by $\tau_i^G(g)f(x) = f(g^{-1}x)$ for $f \in L^2(\mathcal{O}_i), x \in \mathcal{O}_i$) with the following property: U maps eigenfunctions on $\mathcal{O}_1$ to eigenfunctions on $\mathcal{O}_2$ associated with the same eigenvalue.*

Then $\mathcal{O}_1/G$ and $\mathcal{O}_2/G$, equipped with the orbifold structure from Theorem 2.2.4 and the induced Riemannian metric from Corollary 2.3.18 are isospectral orbifolds.

Proof. We just adapt the proof of [Sut06] Theorem 3.15 to this very simple case. First note that since for $i = 1, 2$ the quotient map $\mathcal{O}_i \to \mathcal{O}_i/G$ is a Riemannian orbifold covering, its pullback gives (for $\lambda \geqslant 0$) an isomorphism between the spaces $E_\lambda(\mathcal{O}_i/G)$ and $E_\lambda(\mathcal{O}_i)^G$ (see Definition 3.1.1 and set $E_\lambda = \{0\}$ if λ is not an eigenvalue).

For $i = 1, 2$ restricting the left-regular representation $\tau_i\colon G \to \mathrm{Aut}(L^2(\mathcal{O}_i))$ to $E_\lambda(\mathcal{O}_i)$ gives a representation $\tau_{i,\lambda}\colon G \to \mathrm{Aut}(E_\lambda(\mathcal{O}_i))$. Now note that if $1_G\colon G \to \mathrm{Aut}(\mathbb{R})$ denotes the trivial G-representation, then $\dim(E_\lambda(\mathcal{O}_i)^G)$ is the multiplicity $[\tau_{i,\lambda} : 1_G]$.

Since $\mathcal{O}_1$ and $\mathcal{O}_2$ are equivariantly isospectral with respect to G, there is a unitary equivalence $U\colon L^2(\mathcal{O}_1) \to L^2(\mathcal{O}_2)$ between τ_1 and τ_2 mapping eigenfunctions on $\mathcal{O}_1$ to eigenfunctions on $\mathcal{O}_2$ associated with the same eigenvalue. Thus, restricting U to $E_\lambda(\mathcal{O}_1)$ gives an equivalence between $\tau_{1,\lambda}$ and $\tau_{2,\lambda}$ and hence for every $\lambda \geqslant 0$:

$$\dim(E_\lambda(\mathcal{O}_1/G)) = [\tau_{1,\lambda} : 1_G] = [\tau_{2,\lambda} : 1_G] = \dim(E_\lambda(\mathcal{O}_2/G)).$$

In other words, $\mathcal{O}_1/G$ and $\mathcal{O}_2/G$ are indeed isospectral. □

The orbifolds from Theorem 4.1.1 are then seen to be equivariantly isospectral with respect to the torus T from that Theorem via the same argument as for the manifold version (for which the equivariant isospectrality was already observed in [Sut06]).

Proposition 5.4.2. *Under the conditions of Theorem 4.1.1 the two Riemannian orbifolds $(\mathcal{O}, g)$, $(\mathcal{O}', g')$ are equivariantly isospectral with respect to T.*

Proof. Let $H = H^1(\mathcal{O}, g)$, $H' = H^1(\mathcal{O}', g')$ be the Sobolev spaces as in the proof of Theorem 4.1.1 and let $F\colon H' \to H$ be the L^2-norm preserving isometry from the proof of that theorem. Note that by construction F is T-equivariant. Moreover, for any eigenvalue λ of $\mathcal{O}$

$$E_\lambda(\mathcal{O}) = \{0\} \cup \{f \in H \setminus \{0\};\ R(f) = \lambda \text{ and } f \perp E_\mu(0)\ \forall 0 \leqslant \mu < \lambda\}.$$

Since F preserves Rayleigh quotients, it follows inductively from this characterization that F maps $E_\lambda(\mathcal{O}')$ to $E_\lambda(\mathcal{O})$. Thus, F is a T-equivariant isometry which maps eigenfunctions to eigenfunctions associated with the same eigenvalue. Since F preserves L^2-norms, it extends to an isometry from $L^2(\mathcal{O}')$ to $L^2(\mathcal{O})$ with the same properties. □

Theorem 2.2.4 and Proposition 5.4.2 now imply that in the situation of Theorem 5.2.4 with G a finite subgroup of T the two orbifolds $(\mathcal{O}/G, g_\lambda), (\mathcal{O}/G, g_{\lambda'})$ are isospectral. In particular, the examples in Chapter 5 give isospectral metrics on finite quotients of weighted projective spaces.

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

São Paulo, den 24.03.2010 Martin Weilandt